

DIFFERENTIAL OPERATORS ON THE FREE ALGEBRAS

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ABSTRACT. Following the definitions of the algebras of differential operators, β -differential operators, and the quantum differential operators on a noncommutative (graded) algebra given in [6], we describe these operators on the free associative algebra. We further study their properties.

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1. INTRODUCTION

Let \mathbb{k} denote a field, and R a \mathbb{k} -algebra. Any R -bimodule M under consideration satisfies $\alpha m = m\alpha$ for $m \in M$ and $\alpha \in \mathbb{k}$. In [6] V.A.Lunts and A.L.Rosenberg defined an R -subbimodule, *the differential part of M* , denoted by M_{diff} . The R -bimodule M_{diff} has a filtration $M_0 \subset M_1 \subset \dots$, where M_i is called *the i -th differential part of M* . In particular, the elements of the differential part of the R -bimodule $\text{Hom}_{\mathbb{k}}(R, R)$ are called *the differential operators on R* . This R -bimodule is an algebra and is denoted by $D_{\mathbb{k}}(R)$. The differential operators on R of order $\leq n$ are elements of the n -th differential part of $\text{Hom}_{\mathbb{k}}(R, R)$. Analogously, β -differential operators for any Γ -graded \mathbb{k} algebras (Γ is an abelian group) are defined in [6]. In the particular case of $\Gamma = \mathbb{Z}_2$, the β -differential operators are the super differential operators. In general, under certain conditions on β , the β -differential operators are the coloured differential operators on coloured algebras. A further notion of quantum differential operators on a Γ -graded \mathbb{k} algebra is defined, which allows for viewing action of a quantum group on a ring through its Hopf structure to be via quantum differential operators.

The differential operators on the polynomial algebra and the supercommutative free algebra over fields of characteristic 0 have been widely studied. Smith in [8] studied differential operators on the affine and projective lines for nonzero characteristic. In particular, the algebra of differential operators in the case of nonzero characteristic is not finitely generated.

In this paper we investigate the algebra of differential operators on the free algebra and the β and quantum differential operators on the free algebra graded by \mathbb{Z}^n .

The preliminaries are given in the section 2. The algebra of usual differential operators on the free algebras are described and studied in section 3. When R is the free algebra over n variables, x_1, \dots, x_n , the algebra of 0-th order differential

operators, $D_{\mathbb{k}}^0(R)$, is generated by left and right multiplication homomorphisms. The $D_{\mathbb{k}}^0(R)$ -module of the first order differential operators is generated by derivations. Higher order differential operators are defined in the definition 3.1. Unlike the case of polynomial algebras, the first order differential operators do not generate the algebra of differential operators, $D_{\mathbb{k}}(R)$, even when the characteristic of the underlying field is 0. We are able to prove that $D_{\mathbb{k}}(R)$ is a simple algebra. We also describe several properties of these new operators in this section. Of particular interest is the fact that every differential operator can be written as a power series in a unique way (Remark 3.9).

An analogous study of the β -differential operators is pursued in section 4. Since the β -differential operators are similar to the usual differential operators, the proofs of the results are not presented.

Finally, a study of the quantum differential operators on the free algebras is presented in section 5. We present some properties of these quantum differential operators in this section. The algebra of quantum differential operators on the free algebra, $D_q(R)$, has a more complicated structure than $D_{\mathbb{k}}(R)$. A study of the algebra structure of $D_q(R)$ (for example, checking whether it is simple, or a domain) will be pursued in a later project. In this paper, we primarily address the properties of $D_{\mathbb{k}}(R)$.

There are no restrictions on the characteristic of the underlying field unless otherwise mentioned.

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2. PRELIMINARIES

Let \mathbb{k} be a field, and R be an associative, unital \mathbb{k} -algebra. We let \otimes denote $\otimes_{\mathbb{k}}$. Let R^e denote $R \otimes R^o$ where R^o is the opposite ring of R . Let M be an R -bimodule, equivalently, a left R^e -module. We recall the definition of the *differential part* of M , denoted by M_{diff} from [6]. The *centre* of M is the \mathbb{k} -vector space

$$Z(M) = \{m \in M \mid rm = mr \text{ for all } r \in R\}.$$

If R is commutative, then $Z(M)$ is an R^e -submodule of M . The *differential part*, of M is the R -bimodule $M_{diff} = \cup_{i=0}^{\infty} M_i$, where M_i is the R -bimodule generated by $\{m \in M \mid \bar{m} \in Z(M/M_{i-1})\}$ with $M_{-1} = 0$. Each M_i is called the *i-th differential part* of M .

Notation : For $r \in R$ and $m \in M$, let $[m, r] := mr - rm$ and $[r, m] := rm - mr$. For any \mathbb{k} -algebra A , we set $[a, b] := ab - ba$ for $a, b \in A$.

The vector space $Hom_{\mathbb{k}}(R, R)$ is an R -bimodule as follows: For $r, s \in R$ and $\varphi \in Hom_{\mathbb{k}}(R, R)$, we let $(r\varphi)(s) = r\varphi(s)$, and $(\varphi r)(s) = \varphi(rs)$.

The R -bimodule $D_{\mathbb{k}}^m(R)$ of differential operators on R of order m is the m -th differential part of $\text{Hom}_{\mathbb{k}}(R, R)$. Note that, if R is commutative, a homomorphism $\varphi \in \text{Hom}_{\mathbb{k}}(R, R)$ is said to be a differential operator of order m if $[\cdots [[\varphi, r_0], r_1], \cdots, r_m] = 0$ for all $r_i \in R$.

We see that $\varphi \in Z(\text{Hom}_{\mathbb{k}}(R, R))$ implies, $\varphi(r) = (\varphi r)(1) = (r\varphi)(1) = r\varphi(1)$. For any $r \in R$, let $\lambda_r, \rho_r \in \text{Hom}(R, R)$ be homomorphisms given by $\lambda_r(s) = rs$ and $\rho_r(s) = sr$ for $r \in R$. That is, $\varphi = \rho_{\varphi(1)}$, the homomorphism given by right multiplication by $\varphi(1)$. Thus, $D_{\mathbb{k}}^0(R)$ is the algebra generated by the set $\{\lambda_r, \rho_r | r \in R\}$. There is a surjection $R \otimes_{Z(R)} R^o \twoheadrightarrow D_{\mathbb{k}}^0(R)$ given by $a \otimes b^o \mapsto \lambda_a \rho_b$ for $a, b \in R$ where $Z(R)$ is the centre of R .

By Remark 1.1.2.8 of [6], we have $D_{\mathbb{k}}^i(R)D_{\mathbb{k}}^j(R) \subset D_{\mathbb{k}}^{i+j}(R)$. Hence, $D_{\mathbb{k}}^0(R)$ is a ring, called the ring of inner differential operators which contains R seen as λ_r for $r \in R$. The ring of differential operators $D_{\mathbb{k}}(R)$ is the union of $D_{\mathbb{k}}^i(R)$.

If φ is a derivation, then $\varphi(rs) = r\varphi(s) + \varphi(r)s$. Thus, $[\varphi, r] = \lambda_{\varphi(r)}$. Hence, a derivation is a first order differential operator.

For I be a two sided ideal of a \mathbb{k} -algebra R let

$$\mathcal{S}_I = \{\varphi \in D_{\mathbb{k}}(R) \mid \varphi(I) \subset I\} \quad \text{and} \quad \mathcal{J}_I = \{\varphi \in D_{\mathbb{k}}(R) \mid \varphi(R) \subset I\}.$$

Then \mathcal{S}_I is a filtered subalgebra of $D_{\mathbb{k}}(R)$ with \mathcal{J}_I as a filtered ideal.

Proposition 2.1. *The natural map of \mathbb{k} -algebras $\mathcal{S}_I/\mathcal{J}_I \rightarrow \text{Hom}_{\mathbb{k}}(R/I, R/I)$ gives a map of filtered \mathbb{k} -algebras $\mathcal{S}_I/\mathcal{J}_I \rightarrow D_{\mathbb{k}}(R/I)$.*

Proof. For $\varphi \in \text{Hom}_{\mathbb{k}}(R, R)$, with $\varphi(I) \subset I$ let $\overline{\varphi}$ denote the corresponding map on R/I . Let \overline{a} denote the image of $a \in R$. Note, $\overline{\lambda_a} = \lambda_{\overline{a}}$, $\overline{\rho_a} = \rho_{\overline{a}}$, and $[\overline{\varphi}, \overline{a}] = [\overline{\varphi}, \overline{a}]$. The claim then follows. \square

We now refer to [6] and present some preliminaries on beta-differential operators which are generalizations of superderivations and their higher powers, see [5]. Let Γ be an abelian group. Fix a bicharacter $\beta : \Gamma \times \Gamma \longrightarrow \mathbb{k}^*$. Let R be a Γ -graded \mathbb{k} -algebra and M a Γ -graded R -bimodule. Let $Z_{\beta}(M)$ denote the β -center of M defined as the \mathbb{k} -span of homogeneous elements $m \in M$ such that

$$mr = \beta(d_m, d_r)rm \quad \text{for any homogeneous } r \in R,$$

where d_x denotes the degree of x . The β -differential part of M is the R -bimodule $M_{\beta} = \cup_{i=0}^{\infty} M_{\beta,i}$, where $M_{\beta,i}$ is the R -bimodule generated by the set $\{m \in M \mid \bar{m} \in Z_{\beta}(M/M_{\beta,i-1})\}$ with $M_{\beta,-1} = 0$. Each $M_{\beta,i}$ is the i -th β -differential part of M .

Notations: For $r \in R$ and $m \in M$ homogeneous, let $[m, r]_{\beta} := mr - \beta(d_m, d_r)rm$ and $[r, m]_{\beta} := rm - \beta(d_r, d_m)mr$.

For any Γ -graded \mathbb{k} -algebra A , we set $[a, b]_{\beta} := ab - \beta(d_a, d_b)ba$ for homogeneous $a, b \in A$ and extend $[\cdot, \cdot]$ linearly. Note,

$$[ab, c]_{\beta} = \beta(d_b, d_c)[a, c]_{\beta}b + a[b, c]_{\beta} \quad \text{and} \quad [c, ab]_{\beta} = [c, a]_{\beta}b + \beta(d_c, d_a)a[c, b]_{\beta}.$$

The Γ -graded R -bimodule $D_{\beta}^m(R)$ is the m -th β -differential part of the Γ -graded R -bimodule $\text{grHom}_{\mathbb{k}}(R, R)$. An element of $D_{\beta}^m(R)$ is called, the β -differential operator

of order m . One can see that $D_\beta^i(R)D_\beta^j(R) \subset D_\beta^{i+j}(R)$. That is, $D_\beta(R)$ is a filtered \mathbb{k} -algebra.

For a homogeneous $r \in R$, let $\rho_r^\beta \in \text{grHom}_\mathbb{k}(R, R)$ be defined by

$$\rho_r^\beta(s) = \beta(d_r, d_s)sr.$$

We see that $\varphi \in Z_\beta(\text{grHom}_\mathbb{k}(R, R))$ implies, $\varphi(r) = (\varphi r)(1) = \beta(d_\varphi, d_r)r\varphi(1)$ for any homogeneous $r \in R$. Thus, $\varphi = \rho_{\varphi(1)}^\beta$. Thus, the R -bimodule $D_\beta^0(R)$ is generated by the homomorphisms λ_r, ρ_r^β , for $r \in R$, r homogeneous.

We say that a homogenous $\varphi \in \text{grHom}_\mathbb{k}(R, R)$ is a left β -derivation if

$$\varphi(rs) = \varphi(r)s + \beta(d_\varphi, d_r)r\varphi(s)$$

for homogeneous $r \in R$, and any $s \in R$. Note, a left β -derivation is a β -differential operator of order 1. We see that if φ is a homogeneous left β -derivation, then $\forall \gamma_1, \gamma_2 \in \Gamma$,

$$\begin{aligned} \varphi([a, b]_\beta) &= [\varphi(a), b]_\beta + \beta(d_\varphi, d_a)(a\varphi(b) - \beta(d_\varphi, d_a)^{-1}\beta(d_a, d_b)\varphi(b)a); \\ &= [\varphi(a), b]_\beta + \beta(d_\varphi, d_a)[a, \varphi(b)]_\beta \quad \text{if } \beta(\gamma_1, \gamma_2)\beta(\gamma_2, \gamma_1) = 1. \end{aligned}$$

Given two Γ -graded algebras A, B , the vector space $A \otimes B$ is a Γ -graded \mathbb{k} -algebra. We denote by $A \otimes^\beta B$ the set $A \otimes B$, with a multiplication operation which reflects the Γ grading. For $a, c \in A, u, v \in B$ with b, u homogeneous, we have

$$(a \otimes^\beta b)(u \otimes^\beta v) = \beta(d_b, d_u)(au \otimes^\beta bv).$$

This multiplication also makes $A \otimes^\beta B$ into a Γ -graded \mathbb{k} -algebra. We always use this multiplication when we work with tensor product of Γ -graded algebras in the sections of β -differential operators.

Given a Γ -graded algebra R , its β -opposite algebra is denoted by $R^{\beta, o}$, which as a set is $R^{\beta, o} = \{r^o \mid r \in R\}$ and the operations are $r^o + s^o = (r + s)^o$ (addition), $-r^o = (-r)^o$ (additive inverse), and $r^o s^o = \beta(d_r, d_s)(sr)^o$ (multiplication for homogeneous elements).

Using these conventions we see that there is a surjection of Γ -graded \mathbb{k} -algebras $R \otimes^\beta R^{\beta, o} \rightarrow D_\beta^0(R)$ given by $a \otimes^\beta b^o \mapsto \lambda_a \rho_b^\beta$. This surjection reduces to a surjection of \mathbb{k} -algebras $R \otimes_{Z_\beta(R)}^\beta R^{\beta, o} \rightarrow D_\beta^0(R)$ where $Z_\beta(R)$ is the β -centre of R ; that is, it is the subalgebra of R given by $Z_\beta(R) = \{r \in R \mid \lambda_r = \rho_r^\beta\}$.

For I a two sided Γ -graded ideal of R let

$$\mathcal{S}_I^\beta = \{\varphi \in D_\beta(R) \mid \varphi(I) \subset I\} \quad \text{and} \quad \mathcal{J}_I^\beta = \{\varphi \in D_\beta(R) \mid \varphi(R) \subset I\}.$$

Then \mathcal{S}_I^β is a Γ -graded \mathbb{Z} -filtered subalgebra of $D_\beta(R)$ with \mathcal{J}_I^β as a Γ -graded \mathbb{Z} -filtered ideal. The proof of the following is similar to that of Proposition 2.1.

Proposition 2.2. *The natural map $\mathcal{S}_I^\beta / \mathcal{J}_I^\beta \rightarrow \text{grHom}_\mathbb{k}(R/I, R/I)$ of Γ -graded \mathbb{k} -algebras gives a map of Γ -graded \mathbb{Z} -filtered \mathbb{k} -algebras $\mathcal{S}_I^\beta / \mathcal{J}_I^\beta \rightarrow D_\beta(R/I)$.*

Note that if $\beta \equiv 1$, then we get the usual differential operators.

Let $\mathbb{Z}_2 := \mathbb{Z}/2$. A \mathbb{Z}_2 -graded \mathbb{k} -algebra $R = R_0 \oplus R_1$ is called a *superalgebra*. Elements of R_0 are called *even* and those of R_1 are called *odd*. For a homogeneous element $a \in R$, we let $p(a)$ denote its *parity* (which is the same as *degree* in this case). For a superalgebra R , we define a bicharacter $\beta : \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{k}^*$ by setting $\beta(x, y) = (-1)^{xy}$ and study the β -differential operators on a superalgebra R . The superdifferential operators (respectively, the superderivations) are special cases of β -differential operators (respectively, the β -derivations) on a superalgebra. More generally, when $\beta(a, b)\beta(b, a) = 1$, we get the notion of coloured differential operators. Note that most of the existing studies of super-differential operators (respectively, coloured differential operators) are on supercommutative (respectively, coloured-commutative) algebras. In this article we study the β -differential operators on the free algebra on several variables.

We now recall the definition of the algebra of quantum differential operators ([6]). Let Γ be an abelian group. Fix a bicharacter $\beta : \Gamma \times \Gamma \longrightarrow \mathbb{k}^*$.

Let R be a Γ -graded \mathbb{k} -algebra and M a Γ -graded R -bimodule.

Let $\mathcal{Z}_q(M)$ denote the *quantum-center* of M defined as the \mathbb{k} -span of homogeneous elements $m \in M$ for which there exists a $d \in \Gamma$ such that

$$mr = \beta(d, d_r)rm \text{ for any homogeneous } r \in R.$$

For each $a \in \Gamma$, define $\sigma_a \in \text{grHom}_{\mathbb{k}}(M, M)$ defined by $\sigma_a(m) = \beta(a, d_m)m$ for homogeneous $m \in M$, and extend σ_a linearly on all of M . For $m \in M, r \in R$, let $[m, r]_a = mr - \sigma_a(r)m$. Using these notations,

$$\mathcal{Z}_q(M) = \mathbb{k} - \text{span}\{\text{homogeneous } m \mid \exists a \in \Gamma \text{ such that } [m, r]_a = 0 \forall r \in R\}.$$

Let $M_{q,0} = R\mathcal{Z}_q(M)R$. For $i \geq 1$, $M_{q,i}$ denotes the R -bimodule generated by the set

$$\mathbb{k} - \text{span}\{\text{homogeneous } m \mid \exists a \in \Gamma \text{ such that } [m, r]_a \in M_{q,i-1} \forall r \in R\}.$$

Note, $M_{q,0} \subset M_{q,1} \subset \dots$ and $M_{q-\text{diff}} = \cup_{i \geq 0} M_{q,i}$. When $M = \text{grHom}_{\mathbb{k}}(R, R)$ we get the filtered algebra of quantum differential operators $D_q(R) = M_{q-\text{diff}}$ and the R -bimodule of quantum differential operators of order $\leq i$ is $D_q^i(R) = M_{q,i}$.

The algebra $D_q^0(R)$ is generated by the set $\{\lambda_r, \rho_s, \sigma_a \mid r, s \in R, a \in \Gamma\}$ where

$$\lambda_r \rho_s = \rho_s \lambda_r, \sigma_a \lambda_r = \lambda_{\sigma_a(r)} \sigma_a, \text{ and } \sigma_a \rho_r = \rho_{\sigma_a(r)} \sigma_a.$$

Given a ring S , and a group G of automorphisms of S , denote by $S \# G$ the skew group ring on S by G ; that is, $S \# G$ is a free left S -module, $\oplus_{g \in G} Sg$, with basis G , and with multiplication given by

$$(r_1 g_1)(r_2 g_2) = r_1 g_1(r_2) g_1 g_2 \quad r_1, r_2 \in S, g_1, g_2 \in G.$$

Now Γ acts on R via automorphisms σ_a for $a \in \Gamma$. We thus get a surjection

$$(R \otimes_{Z(R)} R^o) \# \Gamma \rightarrow D_q^0(R) \quad \text{given by } (a \otimes b^o) \gamma \mapsto \lambda_a \rho_b \sigma_\gamma.$$

For $i \geq 1$, each $D_q^i(R)$ is the R -bimodule generated by the \mathbb{k} -span of the set

$$\{\text{homogeneous } \varphi \mid \exists a \in \Gamma \text{ such that } [\varphi, r]_a \in D_q^{i-1}(R)\}.$$

Equivalently, it has been shown in [3] that each $D_q^i(R)$ is the $D_q^0(R)$ -bimodule generated by the \mathbb{k} -span of the set

$$\{\text{homogeneous } \varphi \mid [\varphi, r] \in D_q^{i-1}(R)\}.$$

For each $a \in \Gamma$, let $\varphi \in \text{grHom}_{\mathbb{k}}(R, R)$ be a left skew σ_a -derivation. That is, $\varphi(rs) = \varphi(r)s + \sigma_a(r)\varphi(s) \forall r, s \in R$. Then $[\varphi, r]_a = \lambda_{\varphi(r)} \in D_q^0(R), \forall r \in R$. That is, $\varphi \in D_q^1(R)$. Similarly, for each $a \in \Gamma$, let $\psi \in \text{grHom}_{\mathbb{k}}(R, R)$ be a right skew σ_a -derivation; that is, $\psi(rs) = \psi(r)\sigma_a(s) + r\psi(s) \forall r, s \in R$. Then, $\forall r \in R$, $[\psi, r] = \lambda_{\psi(r)}\sigma_a \in D_q^0(R)$. Note that φ is a left skew σ_a -derivation, if and only if $\varphi\sigma_{-a}$ is a right skew σ_{-a} -derivation.

For I be a two sided Γ -graded ideal of R let

$$\mathcal{S}_I^q = \{\varphi \in D_q(R) \mid \varphi(I) \subset I\} \quad \text{and} \quad \mathcal{J}_I^q = \{\varphi \in D_q(R) \mid \varphi(R) \subset I\}.$$

Then \mathcal{S}_I^q is a Γ -graded \mathbb{Z} -filtered subalgebra of $D_q(R)$ with \mathcal{J}_I^q as a Γ -graded \mathbb{Z} -filtered ideal. Again,

Proposition 2.3. *The natural map $\mathcal{S}_I^q/\mathcal{J}_I^q \rightarrow \text{grHom}_{\mathbb{k}}(R/I, R/I)$ of Γ -graded \mathbb{k} -algebras gives a map of Γ -graded \mathbb{Z} -filtered \mathbb{k} -algebras $\mathcal{S}_I^q/\mathcal{J}_I^q \rightarrow D_q(R/I)$.*

Again, if $\beta \equiv 1$, then we get the usual differential operators.

3. THE FREE ALGEBRA.

Let $R = \mathbb{k}\langle x_1, \dots, x_n \rangle$ be the free algebra over \mathbb{k} generated by x_1, \dots, x_n . When $n = 1$, R is the polynomial ring in one variable, and the differential operators on polynomial rings have been well studied. Therefore, **assume that** $n > 1$.

The ring R has only associative relations. It is a domain, and the monomials in the x_i are independent over \mathbb{k} , so we should expect to find few relations on $D_{\mathbb{k}}^0(R)$. For the proof, we will need the following

Lemma 3.1. *For $i = 1, \dots, m$, let a_i and b_i be monic monomials. Let $d_i = \deg(a_i)$ and $d = \max\{d_i\}$. Then $a_i x_1^d x_2 b_i = a_j x_1^d x_2 b_j$ if and only if $a_i = a_j$ and $b_i = b_j$.*

Proof. If $a_i = a_j$ and $b_i = b_j$, then clearly $a_i x_1^d x_2 b_i = a_j x_1^d x_2 b_j$.

Assume $d_i \leq d_j$. Then the first d_i terms of a_i and a_j coincide. Thus there is some monic monomial c such that $\deg(c) = d_j - d_i$ and $a_j = a_i c$. Hence $x_1^d x_2 b_i = c x_1^d x_2 b_j$.

Since $\deg(c) \leq d$, we have $c = x_1^{d_j - d_i}$. Cancelling x_1^d yields $x_2 b_i = x_1^{d_j - d_i} x_2 b_j$. Thus $d_i = d_j$ and $a_i = a_j$. Cancelling x_2 yields $b_i = b_j$. \square

Proposition 3.1. *The associative algebra $R \otimes R^o$ and $D_{\mathbb{k}}^0(R)$ are isomorphic.*

Proof. The centre of R is just \mathbb{k} , and thus we have a surjection $R \otimes R^o \twoheadrightarrow D_{\mathbb{k}}^0(R)$ given by $a \otimes b^o \mapsto \lambda_a \rho_b$ for $a, b \in R$. It remains to show that this surjection is injective.

An element t of $R \otimes R^o$ may be written as $t = \sum \alpha_i a_i \otimes b_i$ where $\alpha_i \in \mathbb{k}$ and a_i and b_i are monic monomials. Then $t \mapsto \phi_t = \sum \alpha_i \lambda_{a_i} \rho_{b_i}$. Suppose $\phi_t = 0$. Let $d_i = \deg(a_i)$ and $d = \max\{d_i\}$. Then by the lemma, $\{a_i x_1^d x_2 b_i\}$ are pairwise

independent. Thus no two terms of $\phi_t(x_1^d x_2)$ can cancel. As R is the free algebra, $\phi_t(x_1^d x_2)$ is 0 if and only if all $\alpha_i = 0$. That is, $\phi_t(x_1^d x_2) = 0$ if and only if $t = 0$. \square

Corollary 3.1. *The Gelfand-Kirillov dimension of $D_{\mathbb{k}}^0(R)$ (and hence of $D_{\mathbb{k}}(R)$) is infinity.*

Proof. The Gelfand-Kirillov dimension of the free algebra R is infinity. \square

Corollary 3.2. *The centre of $D_{\mathbb{k}}(R)$ is \mathbb{k} .*

Proof. Let $\varphi \in D_{\mathbb{k}}(R)$ be such that $[\varphi, \psi] = 0$ for any $\psi \in D_{\mathbb{k}}(R)$. In particular, $[\varphi, x_i] = [\varphi, \lambda_{x_i}] = 0$ for all $i \leq n$. That is, $\varphi = \rho_a$ for some a in the centre of R . Hence, $a \in \mathbb{k}$. \square

Proposition 3.2. *A derivation of R which is in $D_{\mathbb{k}}^0(R)$ is a sum of inner derivations.*

Proof. Let ϕ be in $D_{\mathbb{k}}^0(R)$ be a derivation. Since $D_{\mathbb{k}}^0(R)$ is generated by left and right multiplications, we can write

$$\phi = \sum \alpha_i \lambda_{a_i} \rho_{b_i}$$

We may assume that if $i \neq j$, then $(a_i, b_i) \neq (a_j, b_j)$.

Since ϕ is a derivation, it has no constant term, so a_i and b_i are not both 1.

Let $d = \max\{\deg(a_i) + \deg(b_i)\}$ and $\tau = x_1^d x_2$. Then $\phi(\tau^2) = \sum \alpha_i a_i \tau^2 b_i$. Since ϕ is a derivation, we also can write $\phi(\tau^2) = \sum \alpha_i (a_i \tau b_i \tau + \tau a_i \tau b_i)$.

The monomials of this expression have three forms. Put $A_i = a_i \tau^2 b_i$, $B_i = a_i \tau b_i \tau$, and $C_i = \tau a_i \tau b_i$. Then $\sum \alpha_i A_i = \sum \alpha_i (B_i + C_i)$. Comparing terms, we have three possible relations among the monomials.

If $A_i = B_j$, then $\tau b_i = b_j \tau$. Thus b_j is a left factor of τ and τ is a left factor of $b_j \tau$. Thus $b_j = 1$. It follows that $b_i = 1$ and $a_i = a_j$. Because the (a_i, b_i) are chosen to be pairwise distinct, we have $i = j$.

If $A_i = C_j$, then $a_i \tau = \tau a_j$. Again, we see $a_i = 1$, and so $a_j = 1$, $b_i = b_j$, and $i = j$.

If $B_i = C_j$, then $a_i \tau b_i \tau = \tau a_j \tau b_j$. Thus $a_i = 1$, $b_i = a_j$, and $b_j = 1$.

In particular, any non-zero term of $\phi(\tau^2)$ must have either $a_i = 1$ or $b_i = 1$. Let $I = \{i \mid b_i = 1\}$ and $J = \{i \mid a_i = 1\}$. Since ϕ has no constant terms, I and J do not intersect. Then $\sum_{i \in I} \alpha_i A_i = \sum_{i \in I} \alpha_i B_i$ and $\sum_{j \in J} \alpha_j A_j = \sum_{j \in J} \alpha_j C_j$. Thus $0 = \sum_{i \in I} \alpha_i C_i + \sum_{j \in J} \alpha_j B_j$. It follows that for each $i \in I$ there is some $j \in J$ such that $\alpha_i = -\alpha_j$, $a_i = b_j$, and $b_i = a_j = 1$. Let us denote such a j by $j(i)$.

Thus, $\phi = \sum_I \alpha_i (\lambda_{a_i} - \rho_{b_{j(i)}}) = \sum_I \alpha_i (\lambda_{a_i} - \rho_{a_i})$, and so ϕ is a sum of inner derivations. \square

For each $a \in R$, and $i \leq n$ let ∂_i^a be the derivation on R defined by $\partial_i^a(x_j) = \delta_{i,j} a$.

Remark 3.1. *Note, for $a \in R$, the inner-derivation $\lambda_a - \rho_a = \sum_{i=1}^n \partial_i^{a x_i - x_i a}$.*

Proposition 3.3. *We have the following relations among the operators $\partial_i^a, \lambda_b, \rho_b$ for $a, b \in R, i \leq n$.*

$$[\partial_i^a, \lambda_b] = \lambda_{\partial_i^a(b)}, \quad [\partial_i^a, \rho_b] = \rho_{\partial_i^a(b)} \quad [\partial_i^a, \partial_j^b] = \partial_j^{\partial_i^a(b)} - \partial_i^{\partial_j^b(a)}.$$

Proof. Follows from the definition of ∂_i^a for $i \leq n, a \in R$. \square

Remark 3.2. *One can immediately see that the Lie algebra of derivations on R , denoted $\text{Der}(R)$, is not simple even in the case when characteristic of \mathbb{k} is 0. Let $\bar{R} = \mathbb{k}[t_1, \dots, t_n]$ denote the polynomial algebra on n variables. The Lie algebra of derivations on \bar{R} , denoted by $\text{Der}(\bar{R})$ is the vector space spanned by $\{f\partial_{t_i} \mid f \in \bar{R}, 1 \leq i \leq n\}$ where each ∂_{t_i} denotes the usual partial derivation with respect to t_i . Consider the quotient algebra map $\bar{\cdot} : R \rightarrow \bar{R}$ given by $\bar{x}_i = t_i$ for $1 \leq i \leq n$. This map gives rise to a map of Lie algebras $\bar{\cdot} : \text{Der}(R) \rightarrow \text{Der}(\bar{R})$ given by $\bar{\partial}_i^a = \bar{a}\partial_{t_i}$. Note that the adjoint derivations are in the kernel of this map.*

Definition 3.1. *For $r = 1, I = (i_1)$ and $J = (a_1)$, with $i_1 \leq n, a_1 \in R$, set $\partial_I^J = \partial_{i_1}^{a_1}$. For an $r \in \mathbb{N}, r \geq 2$, let $I = (i_1, i_2, \dots, i_r)$ be a sequence of natural numbers $i_j \leq n$ and $J = (a_1, \dots, a_r)$ be a sequence of elements from R . Further, let $\hat{I} = (i_2, \dots, i_r)$ and $\hat{J} = (a_2, \dots, a_r)$. Denote by $\partial_I^J \in D_{\mathbb{k}}^r(R)$ the operator which satisfies the commutator rules*

$$[\partial_I^J, x_{i_1}] = a_1 \partial_{\hat{I}}^{\hat{J}}, \quad [\partial_I^J, x_l] = 0 \text{ for } l \neq i_1, \text{ and } \partial_I^J(1) = 0.$$

Remark 3.3. *Note that ∂_I^J is not the same operator as $\partial_{i_1}^{a_1} \dots \partial_{i_r}^{a_r}$. For instance,*

$$\begin{aligned} [\partial_1^{a_1} \partial_1^{a_2}, x_1] &= a_1 \partial_1^{a_2} + \partial_1^{a_1} a_2 = a_1 \partial_1^{a_2} + a_2 \partial_1^{a_1} + \lambda_{\partial_1^{a_1}(a_2)} \text{ and} \\ [\partial_1^{a_1} \partial_1^{a_2}, x_i] &= 0 \text{ for } i \neq 1. \end{aligned}$$

Therefore, $\partial_1^{a_1} \partial_1^{a_2} = \partial_{(1,1)}^{(a_1, a_2)} + \partial_{(1,1)}^{(a_2, a_1)} + \partial_1^{\partial_1^{a_1}(a_2)}$. In general,

$$\partial_i^{a_1} \partial_j^{a_2} = \partial_{(i,j)}^{(a_1, a_2)} + \partial_{(j,i)}^{(a_2, a_1)} + \partial_j^{\partial_i^{a_1}(a_2)}.$$

Indeed, for $i = j$, the argument is identical to the one given above. For $i \neq j$, note that

$$\begin{aligned} [\partial_i^{a_1} \partial_j^{a_2}, x_i] &= a_1 \partial_j^{a_2}, \\ [\partial_i^{a_1} \partial_j^{a_2}, x_j] &= \partial_i^{a_1} a_2 = a_2 \partial_i^{a_1} + \lambda_{\partial_i^{a_1}(a_2)}, \text{ and} \\ [\partial_i^{a_1} \partial_j^{a_2}, x_k] &= 0 \text{ for } k \neq i, j. \end{aligned}$$

The operator ∂_I^J is so defined that $\partial_I^J(x_{i_1} \dots x_{i_r}) = a_1 \dots a_r$ for $I = (i_1, \dots, i_r)$ and $J = (a_1, \dots, a_r)$. This can be checked by induction on r .

By induction on r , we see that $\partial_{(i,i,\dots,i)}^{(a,a,\dots,a)} = \frac{(\partial_i^a)^r}{r!}$ for any $a \in R$, and $i \leq n$ when the characteristic of \mathbb{k} is 0.

Let $\Delta(R)$ be the associative subalgebra of $D_{\mathbb{k}}(R)$ generated by $D_{\mathbb{k}}^1(R)$ and

$$\Delta^r(R) = D_{\mathbb{k}}^r(R) \cap \Delta(R) \text{ for } r \geq 1.$$

Since $\rho_a = \lambda_a - \sum_{i=1}^n \partial_i^{[a, x_i]}$, the algebra $\Delta(R)$ is generated by the set $\{\lambda_{x_i}, \partial_i^w \mid i \leq n, w \text{ word} \in R\}$. If $\partial_{(1,2)}^{(1,1)} \in \Delta(R)$, then $\partial_{(1,2)}^{(1,1)} = \lambda_a + \sum f_{i_1, \dots, i_r}^{a_1, \dots, a_r} \partial_{i_1}^{a_1} \dots \partial_{i_r}^{a_r}$ for $f_{i_1, \dots, i_r}^{a_1, \dots, a_r}, a \in R$, and words $a_1, \dots, a_r \in R$. Since $\partial_{(1,2)}^{(1,1)}(1) = 0$, we get $a = 0$. Let $c = x_1 x_2 - x_2 x_1$. Note that $\partial_{(1,2)}^{(1,1)}(c) = 1$. For any derivation φ , we see that $\varphi(c) = [\varphi(x_1), x_2] + [x_1, \varphi(x_2)]$. In other words, either $\varphi(c) = 0$, or is of degree greater than 1. Thus we get a contradiction. Therefore $\partial_{(1,2)}^{(1,1)} \notin \Delta(R)$. Hence $\Delta(R) \neq D_{\mathbb{k}}(R)$. But we have the following theorem.

Theorem 3.1. For any sequences $I = (i_1, \dots, i_r)$ and $J = (a_1, \dots, a_r)$ with $1 \leq i_s \leq n$ for and $a_s \in R$,

$$\sum_{\sigma \in S_r} \partial_{\sigma(I)}^{\sigma(J)} - \partial_{i_1}^{a_1} \partial_{i_2}^{a_2} \dots \partial_{i_r}^{a_r} \in \Delta^{r-1}(R);$$

here S_r denotes the group of permutations over r elements, $\sigma(I) = (i_{\sigma(1)}, \dots, i_{\sigma(r)})$ and $\sigma(J) = (a_{\sigma(1)}, \dots, a_{\sigma(r)})$. That is, $\sum_{\sigma \in S_r} \partial_{\sigma(I)}^{\sigma(J)} \in \Delta^r(R)$.

Proof. From the remark above we see

$$\partial_{i,j}^{a_1, a_2} + \partial_{j,i}^{a_2, a_1} - \partial_i^{a_1} \partial_j^{a_2} = -\partial_j^{\partial_i^{a_1}(a_2)} \in \Delta^1(R).$$

For $i \neq i_1, \dots, i_r$, $\left[\sum_{\sigma \in S_r} \partial_{\sigma(I)}^{\sigma(J)}, x_i \right] = 0$ and so is $[\partial_{i_1}^{a_1} \partial_{i_2}^{a_2} \dots \partial_{i_r}^{a_r}, x_i] = 0$. Now,

$$\left[\sum_{\sigma \in S_r} \partial_{\sigma(I)}^{\sigma(J)}, x_{i_t} \right] = a_t \sum_{\sigma \in S_{r-1}} \partial_{\sigma(\bar{I})}^{\sigma(\bar{J})}$$

where $\bar{I} = (i_1, i_2, \dots, \widehat{i_t}, \dots, i_r)$ and $\bar{J} = (a_1, a_2, \dots, \widehat{a_t}, \dots, a_r)$, where \widehat{b} means that b is absent in the sequence. Further,

$$[\partial_{i_1}^{a_1} \partial_{i_2}^{a_2} \dots \partial_{i_r}^{a_r}, x_{i_t}] = a_t \partial_{i_1}^{a_1} \dots \partial_{i_t}^{\widehat{a_t}} \dots \partial_{i_r}^{a_r} + \psi \text{ for some } \psi \in \Delta^{r-2}(R).$$

Induction completes the proof. \square

Remark 3.4. The algebra Δ is not simple even when the characteristic of \mathbb{k} is 0. Indeed, let \mathcal{I} be the commutator ideal of R ; that is, let \mathcal{I} be the two sided ideal of R generated by the set $\{[a, b] \mid a, b \in R\}$. If we let \bar{a} denote the image of a in R/\mathcal{I} then R/\mathcal{I} is the polynomial algebra in n variables, $\bar{x}_1, \dots, \bar{x}_n$. Note, for $a \in R$ and $i \leq n$, we have

$$\lambda_a(\mathcal{I}) \subset \mathcal{I}, \quad \rho_a(\mathcal{I}) \subset \mathcal{I}, \quad \text{and} \quad \partial_i^a(\mathcal{I}) \subset \mathcal{I}.$$

Hence, $\Delta(\mathcal{I}) \subset \mathcal{I}$. We have an algebra homomorphism

$$\Delta \rightarrow D(R/\mathcal{I}) \text{ given by}$$

$$\lambda_a \mapsto \lambda_{\bar{a}},$$

$$\rho_a \mapsto \rho_{\bar{a}} = \lambda_{\bar{a}},$$

$$\partial_i^a \mapsto \bar{a} \partial_i \quad (\text{here, } \partial_i \text{ denotes the partial derivative on polynomial algebra}).$$

Since $D_{\mathbb{k}}(R/\mathcal{I})$ is generated as an algebra by $D_{\mathbb{k}}^1(R/\mathcal{I})$, this map is surjective with nontrivial kernel (for example, $\lambda_a - \rho_a$ is in the kernel for every $a \in R$).

Moreover, the set $\{\partial_{i_1}^{a_1} \partial_{i_2}^{a_2} \cdots \partial_{i_r}^{a_r}\}_{r \geq 0}$ does not form a free basis of $\Delta(R)$ over R as in the case of polynomial algebra, because $\partial_i^a \partial_j^b - \partial_j^b \partial_i^a - \partial_j^{\partial_i^b(a)} + \partial_i^{\partial_j^a(b)} = 0$. Even if we consider the subset $\{\partial_{i_1}^{a_1} \partial_{i_2}^{a_2} \cdots \partial_{i_r}^{a_r}\}_{r \geq 0, i_1 \leq i_2 \leq \cdots \leq i_r}$ we still do not have a free basis because $\partial_i^a \partial_i^b - \partial_i^b \partial_i^a - \partial_i^{\partial_i^b(a)} + \partial_i^{\partial_i^a(b)} = 0$.

The following two formulae can be checked by induction on r .

$$\begin{aligned} \partial_{i_1}^{a_1} \partial_{i_2}^{a_2} \cdots \partial_{i_r}^{a_r} \lambda_b &= \sum_{s=0}^r \sum_{j_1 < j_2 < \cdots < j_s} \lambda_{\partial_{i_{j_1}}^{a_{j_1}} \partial_{i_{j_2}}^{a_{j_2}} \cdots \partial_{i_{j_s}}^{a_{j_s}}(b)} \prod_{t=1, t \notin \{j_1, \dots, j_s\}}^r \partial_{i_t}^{a_t}; \\ \partial_{i_1}^{a_1} \partial_{i_2}^{a_2} \cdots \partial_{i_r}^{a_r} \rho_b &= \sum_{s=0}^r \sum_{j_1 < j_2 < \cdots < j_s} \rho_{\partial_{i_{j_1}}^{a_{j_1}} \partial_{i_{j_2}}^{a_{j_2}} \cdots \partial_{i_{j_s}}^{a_{j_s}}(b)} \prod_{t=1, t \notin \{j_1, \dots, j_s\}}^r \partial_{i_t}^{a_t}. \end{aligned}$$

Proposition 3.4. *Here are some properties of the operators ∂_I^J :*

- (1) For $r \geq 1$

$$\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}(x_{t_1} \cdots x_{t_r}) = \begin{cases} a_1 \cdots a_r & \text{if } (t_1, \dots, t_r) = (i_1, \dots, i_r) \\ 0 & \text{if } (t_1, \dots, t_r) \neq (i_1, \dots, i_r) \end{cases}$$

Further, $\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}(x_{t_1} \cdots x_{t_k}) = 0$ for $k < r$.

- (2) $[\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}, \lambda_a] = \sum_{j=2}^r \lambda_{\partial_{(i_1, \dots, i_{j-1})}^{(a_1, \dots, a_{j-1})}(a)} \partial_{(i_j, \dots, i_r)}^{(a_j, \dots, a_r)} + \lambda_{\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}(a)}$ for $r \geq 2$.

For $r = 1$, $[\partial_i^a, \lambda_b] = \lambda_{\partial_i^a(b)}$. In particular, $[\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}, \lambda_a] \in D_{\mathbb{k}}^{r-1}(R)$.

- (3) $[\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}, \rho_a] = \sum_{j=2}^r \rho_{\partial_{(i_1, \dots, i_{j-1})}^{(a_1, \dots, a_{j-1})}(a)} \partial_{(i_j, \dots, i_r)}^{(a_j, \dots, a_r)} + \rho_{\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}(a)}$ for $r \geq 2$.

For $r = 1$, $[\partial_i^a, \rho_b] = \rho_{\partial_i^a(b)}$. In particular, $[\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}, \rho_a] \in D_{\mathbb{k}}^{r-1}(R)$.

Proof. (1) Can be checked by induction on r .

- (2) For $r = 1$, the conclusion is in part (2) of proposition 3.3. For $r > 1$ we proceed by induction, assuming the result for earlier cases. Let $I = (i_1, \dots, i_r)$, $J = (a_1, \dots, a_r)$ and $a = x_l b$. Assume the claim for $[\partial_I^J, \lambda_b]$. We now prove the claim for $[\partial_I^J, \lambda_a]$. Consider two cases:

Case $i_1 \neq l$. Here,

$$\begin{aligned} \partial_I^J \lambda_{x_l} \lambda_b &= \lambda_{x_l} (\partial_I^J \lambda_b) \\ &= \lambda_{x_l} \left(\lambda_b \partial_I^J + \sum_{j=2}^r \lambda_{\partial_{(i_1, \dots, i_{j-1})}^{(a_1, \dots, a_{j-1})}(b)} \partial_{(i_j, \dots, i_r)}^{(a_j, \dots, a_r)} + \lambda_{\partial_I^J(b)} \right) \end{aligned}$$

which gives the claim.

Case $i_1 = l$. Here,

$$\begin{aligned}
\partial_I^J \lambda_{x_l} \lambda_b &= \lambda_{x_l} \partial_I^J \lambda_b + \lambda_{a_1} \partial_{(i_2, \dots, i_r)}^{(a_2, \dots, a_r)} \lambda_b \\
&= \lambda_{x_l} \left(\lambda_b \partial_I^J + \sum_{j=2}^r \lambda_{\partial_{(i_1, \dots, i_{j-1})}^{(a_1, \dots, a_{j-1})}(b)} \partial_{(i_j, \dots, i_r)}^{(a_j, \dots, a_r)} + \lambda_{\partial_I^J(b)} \right) + \\
&\quad + \lambda_{a_1} \left(\lambda_b \partial_{(i_2, \dots, i_r)}^{(a_2, \dots, a_r)} + \sum_{j=3}^r \lambda_{\partial_{(i_2, \dots, i_{j-1})}^{(a_2, \dots, a_{j-1})}(b)} \partial_{(i_j, \dots, i_r)}^{(a_j, \dots, a_r)} + \lambda_{\partial_{(i_2, \dots, i_r)}^{(a_2, \dots, a_r)}(b)} \right) \\
&= \lambda_{x_l} b \partial_I^J + \sum_{j=3}^r \lambda_{(x_l \partial_{(i_1, \dots, i_{j-1})}^{(a_1, \dots, a_{j-1})} + a_1 \partial_{(i_2, \dots, i_{j-1})}^{(a_2, \dots, a_{j-1})})(b)} \partial_{(i_j, \dots, i_r)}^{(a_j, \dots, a_r)} \\
&\quad + \lambda_{(x_l \partial_{i_1}^{a_1} + a_1)(b)} \partial_{(i_2, \dots, i_r)}^{(a_2, \dots, a_r)} + \lambda_{x_l \partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}(b) + a_1 \partial_{(i_2, \dots, i_r)}^{(a_2, \dots, a_r)}(b)} \\
&= \lambda_a \partial_I^J + \sum_{j=2}^r \lambda_{\partial_{(i_1, \dots, i_{j-1})}^{(a_1, \dots, a_{j-1})}(a)} \partial_{(i_j, \dots, i_r)}^{(a_j, \dots, a_r)} + \lambda_{\partial_I^J(a)}.
\end{aligned}$$

- (3) We address this by induction on r . For $r = 1$, the conclusion is in part (2) of proposition 3.3. For $r \geq 2$, consider

$$\begin{aligned}
&[\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)} \rho_a - \rho_{\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}(a)}, x_l] = \delta_{i_1, l} \partial_{(i_2, \dots, i_r)}^{(a_2, \dots, a_r)} \rho_a \text{ and} \\
&\left(\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)} \rho_a - \rho_{\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}(a)} \right) (1) = 0.
\end{aligned}$$

Now, by induction assumption,

$$\partial_{(i_2, \dots, i_r)}^{(a_2, \dots, a_r)} \rho_a = \rho_a \partial_{(i_2, \dots, i_r)}^{(a_2, \dots, a_r)} + \sum_{j=3}^r \rho_{\partial_{(i_j, \dots, i_r)}^{(a_j, \dots, a_r)}(a)} \partial_{(i_2, \dots, i_{j-1})}^{(a_2, \dots, a_{j-1})} + \rho_{\partial_{(i_2, \dots, i_r)}^{(a_2, \dots, a_r)}(a)}.$$

Therefore,

$$\begin{aligned}
\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)} \rho_a - \rho_{\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}(a)} &= \rho_a \partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)} + \sum_{j=3}^r \rho_{\partial_{(i_j, \dots, i_r)}^{(a_j, \dots, a_r)}(a)} \partial_{(i_1, \dots, i_{j-1})}^{(a_1, \dots, a_{j-1})} \\
&\quad + \rho_{\partial_{(i_2, \dots, i_r)}^{(a_2, \dots, a_r)}(a)} \partial_{i_1}^{a_1}
\end{aligned}$$

which gives us the claim. \square

Remark 3.5. For $I = (i_1, \dots, i_r)$, $A = (a_1, \dots, a_r)$ and $J = (i_l, \dots, i_{l+t})$ for some $1 \leq l \leq l+t \leq r$, let $A_J = (a_l, \dots, a_{l+t})$. For two finite sequences J, K , by (J, K) we mean the concatenation of J and K . Then Part (1) or (2) of the above proposition gives

$$\partial_I^A(ab) = \sum_{(J, K)=I} \partial_J^{A_J}(a) \partial_K^{A_K}(b).$$

Part (1) of the above Proposition has an immediate important corollary.

Corollary 3.3. *R as a left $D_{\mathbb{k}}(R)$ module is simple.*

Proof. Let any $r \in R, r \neq 0, r \notin \mathbb{k}$. We need to show the existence of $\Phi \in D_{\mathbb{k}}(R)$ such that $\Phi(r) \in \mathbb{k}^*$. Suppose degree of r is n , $n \geq 1$. Let $\alpha_{i_1, \dots, i_l} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_l}^{k_l}$ be a summand of r of degree n . That is,

$$r = \alpha_{i_1, \dots, i_l} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_l}^{k_l} + \text{terms of degree less than or equal to } n.$$

Let $I = \underbrace{(i_1, \dots, i_1)}_{k_1 \text{ times}}, \underbrace{(i_2, \dots, i_2)}_{k_2 \text{ times}}, \dots, \underbrace{(i_l, \dots, i_l)}_{k_l \text{ times}}$ and $J = \underbrace{(1, 1, \dots, 1)}_{n \text{ times}}$. Then

$$\partial_I^J(r) = \alpha_{i_1, \dots, i_l} \in \mathbb{k}^*.$$

□

Let S_{r+s} be the group of permutations of $r+s$ elements, and let $T_{(r,s)}$ be the subset of S_{r+s} which preserves the order of the first r elements and the last s elements. That is, $\tau \in T_{(r,s)}$ is increasing on $1, \dots, r$ and on $r+1, \dots, r+s$. Note that $T_{(0,s)}$ and $T_{(r,0)}$ consist of just the identity permutation.

For any $\sigma \in S_{r+s}$ and k , $1 \leq k \leq r+s$, we can define a permutation $\sigma^{(k)} \in S_{r+s+1}$ such that

$$\sigma^{(k)}(i) = \begin{cases} \sigma(i) + 1 & i < k \\ 1 & i = k \\ \sigma(i-1) + 1 & i > k \end{cases}$$

That is, $\sigma^{(k)}$ takes k to 1 and acts like σ on the remaining symbols. As σ is a bijection from $\{1, \dots, r+s\}$ to $\{1, \dots, r+s\}$, we can see that $\sigma^{(k)}$ is a bijection from $\{1, \dots, \hat{k}, \dots, r+s+1\}$ to $\{2, \dots, r+s+1\}$. Hence $\sigma^{(k)}$ is indeed a permutation.

Lemma 3.2. $T_{(r,s)} = \{\tau^{(1)} \mid \tau \in T_{(r-1,s)}\} \cup \{\tau^{(r+1)} \mid \tau \in T_{(r,s-1)}\}$.

Proof. First, let $\tau \in T_{(r-1,s)}$. We will show $\tau^{(1)} \in T_{(r,s)}$.

As $\tau^{(1)}$ on $2, \dots, r+s$ is just a translation of τ , and τ is increasing on $1, \dots, r-1$ and on $r, \dots, r-1+s$, we have $\tau^{(1)}$ is increasing on $2, \dots, r$ and on $r+1, \dots, r+s$ respectively. Since $\tau^{(1)}(1) = 1$, we see $\tau^{(1)} \in T_{(r,s)}$.

Second, let $\tau \in T_{(r,s-1)}$. We will show $\tau^{(r+1)} \in T_{(r,s)}$. As τ is increasing on $1, \dots, r$, so is $\tau^{(r+1)}$. As τ is increasing on $r+1, \dots, r+s-1$, so $\tau^{(r+1)}$ is increasing on $r+2, \dots, r+s$. As $\tau^{(r+1)}(r+1) = 1 < \tau^{(r+1)}(r+2)$, we have $\tau^{(r+1)} \in T_{(r,s)}$.

This proves $T_{(r,s)} \supseteq \{\tau^{(1)} \mid \tau \in T_{(r-1,s)}\} \cup \{\tau^{(r+1)} \mid \tau \in T_{(r,s-1)}\}$.

Now, let $\tau \in T_{(r,s)}$. Then either $\tau(1) = 1$ or $\tau(r+1) = 1$.

Case $\tau(1) = 1$. Let $\sigma \in S_{r+s-1}$ be such that $\sigma(i) = \tau(i+1) - 1$. Then $\sigma^{(1)} = \tau$. As τ is increasing on $2, \dots, r$ and on $r+1, \dots, r+s$, we have σ is increasing on $1, \dots, r-1$ and on $r, \dots, r-1+s$, respectively. Thus $\sigma \in T_{(r-1,s)}$.

Case $\tau(r+1) = 1$. Let $\sigma \in S_{r+s-1}$ be such that $\sigma(i) = \tau(i) - 1$ for $1 \leq i \leq r$ and $\sigma(i) = \tau(i+1) - 1$ for $r+1 \leq i \leq r+s-1$. Then $\sigma^{(r+1)} = \tau$.

As τ is increasing on $1, \dots, r$ and on $r+2, \dots, r+s$, we have σ is increasing on $1, \dots, r$ and on $r+1, \dots, r+s-1$ respectively. Thus $\sigma \in T_{(r,s-1)}$. □

Recall that (I, J) denotes the concatenation of I with J , and (A, B) denotes the concatenation of A with B .

Theorem 3.2.

$$\partial_I^A \partial_J^B = \sum_{\tau \in T_{(r,s)}} \partial_{\tau((I,J))}^{\tau((A,B))} + \sum \{ \partial_K^C \mid \partial_K^C \text{ is of order less than } r+s \}$$

Proof. Put $N = r + s$. We proceed by induction on N .

Our base case starts with $N = 2$ and $r = s = 1$. Corollary 3.3 proves the result.

Suppose that the theorem holds for $N < N_0$ and that $r + s = N_0$. Then

$$\begin{aligned} \partial_{\hat{I}}^{\hat{A}} \partial_J^B &= \sum_{\tau \in T_{(r-1,s)}} \partial_{\tau((\hat{I},J))}^{\tau((\hat{A},B))} + \sum_{\alpha} \partial_{L_{\alpha}}^{C_{\alpha}} \\ \partial_I^A \partial_{\hat{J}}^{\hat{B}} &= \sum_{\tau \in T_{(r,s-1)}} \partial_{\tau((I,\hat{J}))}^{\tau((A,\hat{B}))} + \sum_{\beta} \partial_{L_{\beta}}^{C_{\beta}} \\ \partial_{\hat{I}}^{\hat{A}} \partial_{\hat{J}}^{\hat{B}} &= \sum_{\tau \in T_{(r-1,s-1)}} \partial_{\tau((\hat{I},\hat{J}))}^{\tau((\hat{A},\hat{B}))} + \sum_{\gamma} \partial_{L_{\gamma}}^{C_{\gamma}} \end{aligned}$$

where $\partial_{L_{\alpha}}^{C_{\alpha}}$, $\partial_{L_{\beta}}^{C_{\beta}}$, and $\partial_{L_{\gamma}}^{C_{\gamma}}$ are all of order $N_0 - 2$ or lower.

For each i , put $f_i = \partial_{i_1}^{a_1}(x_i) = \delta_{ii_1} a_1$, $g_i = \partial_{j_1}^{b_1}(x_i) = \delta_{ij_1} b_1$, and $h_i = \partial_{i_1}^{a_1}(g_i) = \delta_{ij_1} \partial_{i_1}^{a_1}(b_1)$. Then we have

$$\begin{aligned} [\partial_I^A \partial_J^B, \lambda_{x_i}] &= [\partial_I^A, \lambda_{x_i}] \partial_J^B + \partial_I^A [\partial_J^B, \lambda_{x_i}] \\ &= \lambda_{f_i} \partial_{\hat{I}}^{\hat{A}} \partial_J^B + \partial_I^A \lambda_{g_i} \partial_{\hat{J}}^{\hat{B}} \\ &= \lambda_{f_i} \partial_{\hat{I}}^{\hat{A}} \partial_J^B + \lambda_{g_i} \partial_I^A \partial_{\hat{J}}^{\hat{B}} + \lambda_{h_i} \partial_{\hat{I}}^{\hat{A}} \partial_{\hat{J}}^{\hat{B}} \\ &= \sum_{\tau \in T_{(r-1,s)}} \lambda_{f_i} \partial_{\tau((\hat{I},J))}^{\tau((\hat{A},B))} + \sum_{\tau \in T_{(r,s-1)}} \lambda_{g_i} \partial_{\tau((I,\hat{J}))}^{\tau((A,\hat{B}))} \\ &\quad + \sum_{\tau \in T_{(r-1,s-1)}} \lambda_{h_i} \partial_{\tau((\hat{I},\hat{J}))}^{\tau((\hat{A},\hat{B}))} + \sum_{\alpha} \lambda_{f_i} \partial_{L_{\alpha}}^{C_{\alpha}} + \sum_{\beta} \lambda_{g_i} \partial_{L_{\beta}}^{C_{\beta}} + \sum_{\gamma} \lambda_{h_i} \partial_{L_{\gamma}}^{C_{\gamma}} \\ &= \sum_{\tau \in T_{(r-1,s)}} \left[\partial_{(i_1, \tau((\hat{I},J)))}^{(a_1, \tau((\hat{A},B)))}, \lambda_{x_i} \right] + \sum_{\tau \in T_{(r,s-1)}} \left[\partial_{(j_1, \tau((I,\hat{J})))}^{(b_1, \tau((A,\hat{B})))}, \lambda_{x_i} \right] \\ &\quad + \sum_{\tau \in T_{(r-1,s-1)}} \left[\partial_{(j_1, \tau((\hat{I},\hat{J})))}^{(\partial_{i_1}^{a_1}(b_1), \tau((\hat{A},\hat{B})))}, \lambda_{x_i} \right] + \sum_{\alpha} \left[\partial_{(i_1, L_{\alpha})}^{(a_1, C_{\alpha})}, \lambda_{x_i} \right] \\ &\quad + \sum_{\beta} \left[\partial_{(j_1, L_{\beta})}^{(b_1, C_{\beta})}, \lambda_{x_i} \right] + \sum_{\gamma} \left[\partial_{(j_1, L_{\gamma})}^{(\partial_{i_1}^{a_1}(b_1), C_{\gamma})}, \lambda_{x_i} \right] \end{aligned}$$

Note that $(i_1, \tau((\hat{I}, J))) = \tau^{(1)}((I, J))$, $(j_1, \tau((I, \hat{J}))) = \tau^{(r+1)}((I, J))$, $(a_1, \tau((\hat{A}, B))) = \tau^{(1)}((A, B))$, and $(b_1, \tau((A, \hat{B}))) = \tau^{(r+1)}((A, B))$. Thus, by our Lemma,

$$\sum_{\tau \in T_{(r,s)}} \partial_{\tau((I,J))}^{\tau((A,B))} = \sum_{\tau \in T_{(r-1,s)}} \partial_{(i_1, \tau((\hat{I},J)))}^{(a_1, \tau((\hat{A},B)))} + \sum_{\tau \in T_{(r,s-1)}} \partial_{(j_1, \tau((I,\hat{J})))}^{(b_1, \tau((A,\hat{B})))}.$$

Put

$$\begin{aligned} \phi = & \sum_{\tau \in T_{(r,s)}} \partial_{\tau((I,J))}^{\tau((A,B))} + \sum_{\tau \in T_{(r-1,s-1)}} \partial_{(j_1 \tau((\hat{I}, \hat{J})))}^{(\partial_{i_1}^{a_1}(b_1), \tau((\hat{A}, \hat{B})))} + \sum_{\alpha} \partial_{(i_1, L_{\alpha})}^{(a_1, C_{\alpha})} \\ & + \sum_{\beta} \partial_{(j_1, L_{\beta})}^{(b_1, C_{\beta})} + \sum_{\gamma} \partial_{(j_1, L_{\gamma})}^{(\partial_{i_1}^{a_1}(b_1), C_{\gamma})}. \end{aligned}$$

Since $\phi(1) = \partial_I^A \partial_J^B(1) = 0$ and $[\phi, \lambda_{x_i}] = [\partial_I^A \partial_J^B, \lambda_{x_i}]$ for each i , we have $\phi = \partial_I^A \partial_J^B$. Regardless of how the tuples are permuted, $\partial_{(j_1, \hat{I}, \hat{J})}^{(\partial_{i_1}^{a_1}(b_1), \hat{A}, \hat{B})}$ has order $N_0 - 1$.

Likewise, each $\partial_{(i_1, L_{\alpha})}^{(a_1, C_{\alpha})}$, $\partial_{(j_1, L_{\beta})}^{(b_1, C_{\beta})}$, and $\partial_{(j_1, L_{\gamma})}^{(\partial_{i_1}^{a_1}(b_1), C_{\gamma})}$ are of order $N_0 - 1$ or less. Thus our expression ϕ is of the required form. \square

Remark 3.6. *The lower order terms which appear in the statement of Theorem 3.2 can be described. If we denote $\partial_{(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}$ by $\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ i_1 & i_2 & \dots & i_r \end{pmatrix}$ then every lower order term looks like*

$$\begin{pmatrix} a_1 & \dots & a_{p_1-1} & d_1(b_1) & a_{p_{q_1}+1} & \dots & a_{p_{q_2}-1} & d_2(b_2) & \dots \\ i_1 & \dots & i_{p_1-1} & j_1 & a_{p_{q_1}+1} & \dots & a_{p_{q_2}-1} & j_2 & \dots \end{pmatrix}$$

where $d_1(b_1) = \partial_{(i_{p_1}, i_{p_1+1}, \dots, i_{p_{q_1}})}^{(a_{p_1}, a_{p_1+1}, \dots, a_{p_{q_1}})}(b_1)$ and $d_2(b_2) = \partial_{(i_{p_2}, i_{p_2+1}, \dots, i_{p_{q_2}})}^{(a_{p_2}, a_{p_2+1}, \dots, a_{p_{q_2}})}(b_2)$.

The last term is

$$\begin{pmatrix} b_1 & b_2 & \dots & b_{s-1} & \partial_I^A(b_s) \\ j_1 & j_2 & \dots & j_{s-1} & j_s \end{pmatrix}.$$

In particular, the operators which appear in the product $\partial_I^A \partial_J^B$ are of order at least s , which is the order of ∂_J^B . For proof, adapt the proof of Theorem 3.2 along with part (2) of Proposition 3.4. We present a few examples.

$$\begin{aligned} \begin{pmatrix} a \\ i \end{pmatrix} \cdot \begin{pmatrix} b_1 & \dots & b_s \\ j_1 & \dots & j_s \end{pmatrix} = & \sum_{l=1}^s \begin{pmatrix} b_1 & \dots & b_{l-1} & a & b_l & \dots & b_s \\ j_1 & \dots & j_{l-1} & i & j_l & \dots & j_s \end{pmatrix} \\ & + \sum_{l=1}^s \begin{pmatrix} b_1 & \dots & b_{l-1} & \partial_i^a(b_l) & b_{l+1} & \dots & b_s \\ j_1 & \dots & j_{l-1} & j_l & j_{l+1} & \dots & j_s \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} a_1 & \dots & a_r \\ i_1 & \dots & i_r \end{pmatrix} \cdot \begin{pmatrix} b \\ j \end{pmatrix} = & \sum_{l=1}^r \begin{pmatrix} a_1 & \dots & a_{l-1} & b & a_l & \dots & a_r \\ i_1 & \dots & i_{l-1} & j & j_l & \dots & i_r \end{pmatrix} \\ & + \sum_{p=1}^r \sum_{s=0}^{r-p} \begin{pmatrix} a_1 & \dots & a_{p-1} & \partial_{(i_p, \dots, i_{p+s})}^{(a_p, \dots, a_{p+s})}(b) & a_{p+s+1} & \dots & a_r \\ i_1 & \dots & i_{p-1} & j & i_{p+s+1} & \dots & i_r \end{pmatrix}. \end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} a_1 & a_2 \\ i_1 & i_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 & b_2 \\ j_1 & j_2 \end{pmatrix} &= \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ i_1 & i_2 & j_1 & j_2 \end{pmatrix} + \begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ i_1 & j_1 & i_2 & j_2 \end{pmatrix} \\
&+ \begin{pmatrix} b_1 & a_1 & a_2 & b_2 \\ j_1 & i_1 & i_2 & j_2 \end{pmatrix} + \begin{pmatrix} a_1 & b_1 & b_2 & a_2 \\ i_1 & j_1 & j_2 & i_2 \end{pmatrix} \\
&+ \begin{pmatrix} b_1 & a_1 & b_2 & a_2 \\ j_1 & i_1 & j_2 & i_2 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 & a_1 & a_2 \\ j_1 & j_2 & i_1 & i_2 \end{pmatrix} \\
&+ \begin{pmatrix} a_1 & \partial_{i_2}^{a_2}(b_1) & b_2 \\ i_1 & j_1 & j_2 \end{pmatrix} + \begin{pmatrix} a_1 & b_1 & \partial_{i_2}^{a_2}(b_2) \\ i_1 & j_1 & j_2 \end{pmatrix} \\
&+ \begin{pmatrix} b_1 & a_1 & \partial_{i_2}^{a_2}(b_2) \\ j_1 & i_1 & j_2 \end{pmatrix} + \begin{pmatrix} \partial_{i_1}^{a_1}(b_1) & a_2 & b_2 \\ j_1 & i_2 & j_2 \end{pmatrix} \\
&+ \begin{pmatrix} \partial_{i_1}^{a_1}(b_1) & b_2 & a_2 \\ j_1 & j_2 & i_2 \end{pmatrix} + \begin{pmatrix} b_1 & \partial_{i_1}^{a_1}(b_2) & a_2 \\ j_1 & j_2 & i_2 \end{pmatrix} \\
&+ \begin{pmatrix} \partial_{i_1}^{a_1}(b_1) & \partial_{i_2}^{a_2}(b_2) \\ j_1 & j_2 \end{pmatrix} + \begin{pmatrix} b_1 & \partial_{i_1, i_2}^{a_1, a_2}(b_2) \\ j_1 & j_2 \end{pmatrix}.
\end{aligned}$$

Corollary 3.4. $[\partial_I^J, \partial_K^L] \in D_{\mathbb{k}}^{r+t-1}(R)$ for $\partial_I^J \in D_{\mathbb{k}}^r(R)$, $\partial_K^L \in D_{\mathbb{k}}^t(R)$, and $r, t \geq 1$.

Theorem 3.3. For $r \geq 1$ $D_{\mathbb{k}}^r(R)$ is generated as a left $D_{\mathbb{k}}^0(R)$ -module by the set

$$\{\partial_I^J \mid I = (i_1, \dots, i_s), J = (a_1, \dots, a_s), i_j \in \mathbb{N}, i_j \leq n, a_j \in R, 1 \leq s \leq r\} \cup \{\lambda_1\}$$

Proof. Let $\varphi \in D_{\mathbb{k}}^1(R)$ such that $[\varphi, x_i] = \sum_j \lambda_{a_{i,j}} \rho_{b_{i,j}} \in D_{\mathbb{k}}^0(R)$, for $a_{i,j}, b_{i,j} \in R$ and $i \leq n$. Then $\psi = \varphi - \sum_i \sum_j \rho_{b_{i,j}} \partial_i^{a_{i,j}}$ is such that $[\psi, x_i] = 0$ for every $i \leq n$. Therefore $\psi = \rho_r$ for some $r \in R$. Hence, $\varphi = \rho_r + \sum_{i,j} \rho_{b_{i,j}} \partial_i^{a_{i,j}}$.

In general, let $\varphi \in D_{\mathbb{k}}^r(R)$ such that $[\varphi, x_i] = \sum_{i,I,J} \rho_{a_{i,I,J}^i} \lambda_{b_{i,I,J}^i} \partial_I^J \in D_{\mathbb{k}}^{r-1}(R)$, for $a_{i,I,J}^i, b_{i,I,J}^i \in R$, $i \leq n$ and I, J the appropriate sequences of length less than or equal to $r-1$. Then $\psi = \varphi - \sum_{i,I,J} \rho_{a_{i,I,J}^i} \partial_{(i,I)}^{(b_{i,I,J}^i, J)}$ is such that $[\psi, x_i] = 0$ for every $i \leq n$. Therefore $\psi = \rho_s$ for some $s \in R$. Hence, $\varphi = \rho_s + \sum_{i,I,J} \rho_{a_{i,I,J}^i} \partial_{(i,I)}^{(b_{i,I,J}^i, J)}$; here, (i, I) is the concatenation of (i) and I , while $(b_{i,I,J}^i, J)$ is the concatenation of $(b_{i,I,J}^i)$ and J . \square

Suppose $J = (a_1, \dots, a_s)$ is such that $a_p = \alpha_1 b_{p_1} + \alpha_2 b_{p_2}$ for some $p \leq s$, $\alpha_1, \alpha_2 \in \mathbb{k}$, and $b_{p_1}, b_{p_2} \in R$. Then, letting $J_1 = (a_1, \dots, a_{p-1}, b_{p_1}, a_{p+1}, \dots, a_s)$ and $J_2 = (a_1, \dots, a_{p-1}, b_{p_2}, a_{p+1}, \dots, a_s)$ we see by induction on s ,

$$\partial_I^J = \alpha_1 \partial_I^{J_1} + \alpha_2 \partial_I^{J_2}.$$

Thus, $D_{\mathbb{k}}(R)$ is generated as a left $D_{\mathbb{k}}^0(R)$ -module by the set

$$\{\partial_I^J \mid I = (i_1, \dots, i_s), J = (a_1, \dots, a_s), i_j \in \mathbb{N}, \text{ words } a_j \in R, 1 \leq i_j \leq n\} \cup \{\lambda_1\}.$$

Moreover, since for $a \in R$,

$$\lambda_a - \rho_a = \sum_i \partial_i^{[a, x_i]}, \text{ we have } \rho_a = \lambda_a - \sum_i \partial_i^{[a, x_i]}.$$

From Theorem 3.2 and part (2) of Proposition 3.4, we see that $D_{\mathbb{k}}(R)$ as a left R -module (via λ_R) is generated by the set $\{\partial_I^J \mid I = (i_1, \dots, i_s), J = (a_1, \dots, a_s), i_j \in \mathbb{N}, \text{ words } a_j \in R, 1 \leq i_j \leq n\} \cup \{\lambda_1\}$.

In particular, $D_{\mathbb{k}}^1(R)$ is a left R -module generated by $\{\partial_i^a \mid a \text{ is a word in } R\} \cup \{\lambda_1\}$. We suspect that the following proposition is well-known. But we have not found it published anywhere.

Proposition 3.5. *The left R -module $D_{\mathbb{k}}^1(R)$ is free with basis*

$$\{\partial_i^a \mid a \text{ is a word in } R\} \cup \{\lambda_1\}.$$

Proof. Suppose $\varphi = a\lambda_1 + \sum_{i=1}^n \sum_{j=1}^{r_i} h_{ij} \partial_i^{f_{ij}} = 0$ for some $a, h_{ij} \in R$, and f_{ij} words in R . Since $\varphi(1) = 0$, we have, $a = 0$. Hence, $\varphi = \sum_{i=1}^n \sum_{j=1}^{r_i} h_{ij} \partial_i^{f_{ij}} = 0$. Now, for each $i \leq n$ and any word $w \in R$, we have

$$\varphi(wx_i) = \varphi(w)x_i + \sum_{j=1}^{r_i} h_{ij} w f_{ij} = 0.$$

Since $\varphi(w) = 0$, we have $\sum_{j=1}^{r_i} h_{ij} w f_{ij} = 0$ for every word $w \in R$. That is, $\sum_{j=1}^{r_i} h_{ij} \otimes f_{ij}^o \in \text{Ker} : R \otimes R^o \rightarrow D_{\mathbb{k}}^0(R)$ of proposition 3.1. That is, $\sum_{j=1}^{r_i} h_{ij} \otimes f_{ij}^o = 0$, hence the result. \square

Remark 3.7. *Note that this result does not generalize to $D_{\mathbb{k}}^i(R)$ for $i \geq 2$. For example,*

$$\partial_{(1,2)}^{(x_2 x_1, 1)} - \partial_{(1,2)}^{(x_1 x_2, 1)} - x_2 \partial_2^1 + \partial_2^{x_2} = 0.$$

But we can generalize Proposition 3.5 in the following sense.

Proposition 3.6. *For a fixed $s \geq 1$, the set*

$$\{\partial_I^J \mid I = (i_1, \dots, i_s), J = (a_1, \dots, a_s), i_j \leq n, \text{ words } a_j \in R\}$$

generates a free $D_{\mathbb{k}}^0(R)$ -submodule of $D_{\mathbb{k}}^s(R)$.

Proof. The argument is essentially the same as in the proof of Proposition 3.5 and we use induction on s . Suppose $\varphi = \sum_{|I|=s} \sum_k \rho_{q_{I,J}^k} \lambda_{p_{I,J}^k} \partial_I^J = 0$ for some finitely many monomials $q_{I,J}^k, p_{I,J}^k \in R$ with every entry of J a word. For any $t \leq n$, and word $w \in R$, we have $\varphi(wx_t) = \varphi(w)x_t + \sum_{\{I \mid i_s=t\}} \sum_k \rho_{a_s} \rho_{q_{I,J}^k} \lambda_{p_{I,J}^k} \partial_{\hat{I}}^{\hat{J}}(w)$ where $\hat{I} = (i_1, \dots, i_{s-1})$ and $\hat{J} = (a_1, \dots, a_{s-1})$, and we use the fact that $\partial_I^J \rho_{x_t} = \rho_{x_t} \partial_I^J + \rho_{\partial_{i_s}^{a_s}(x_t)} \partial_{\hat{I}}^{\hat{J}}$. Hence, $\varphi = 0$ implies that $\sum_{\{I \mid i_s=t\}} \sum_k \rho_{a_s} \rho_{q_{I,J}^k} \lambda_{p_{I,J}^k} \partial_{\hat{I}}^{\hat{J}} = 0$ and we now appeal to induction to complete the proof. \square

Proposition 3.7. *$D_{\mathbb{k}}(R)$ is spanned as a \mathbb{k} -vector space by the set*

$$\{\partial_I^J \mid I = (i_1, \dots, i_s), J = (a_1, \dots, a_s), i_j \in \mathbb{N}, \text{ words } a_j \in R, 1 \leq i_j \leq n\} \cup \{\lambda_a \mid a \text{ is a word in } R\}.$$

Proof. We already know that $D_{\mathbb{k}}(R)$ as a left R -module (via λ_R) is generated by the set $\{\partial_I^J \mid I = (i_1, \dots, i_s), J = (a_1, \dots, a_s), i_j \in \mathbb{N}, \text{ words } a_j \in R, 1 \leq i_j \leq n\} \cup \{\lambda_1\}$.

Now, let $a \in R$ be a word, $I = (i_1, \dots, i_s)$, $J = (a_1, \dots, a_s)$ for some $s \geq 1$, $i_j \leq n$, and words $a_j \in R$. Let $\hat{I} = (i_2, \dots, i_s)$ and $\hat{J} = (a_2, \dots, a_s)$. Then,

$$[a\partial_I^J, x_{i_1}] = [a, x_{i_1}]\partial_I^J + aa_1\partial_{\hat{I}}^{\hat{J}}, \text{ and}$$

$$[a\partial_I^J, x_p] = [a, x_p]\partial_I^J \text{ for } p \neq i_1.$$

Further, $a\partial_I^J(1) = 0$. That is, letting $(p, I) = (p, i_1, \dots, i_s)$ and $([a, x_p], J) = ([a, x_p], a_1, \dots, a_p)$ for $1 \leq p \leq n$, and $a \cdot J = (aa_1, a_2, \dots, a_s)$ we have

$$a\partial_I^J = \partial_I^{a \cdot J} + \sum_{p=1}^n \partial_{(p, I)}^{([a, x_p], J_p)}.$$

□

Proposition 3.8. *Let $I_1 = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_s)$, $A = (a_1, \dots, a_r)$, $B = (b_1, \dots, b_s)$ with $i_l, j_l \leq n$, and $a_l, b_l \in R$. Let $A \cdot w = (a_1, \dots, a_r w)$ and $w \cdot B = (w b_1, \dots, b_s)$ for $w \in R$. Then*

$$\partial_{(I, J)}^{(A \cdot w, B)} - \partial_{(I, J)}^{(A, w \cdot B)} = \sum_{k=1}^n \partial_{(I, k, J)}^{(A, [w, x_k], B)}.$$

Proof. This is proved by induction on r . Note

$$[\partial_{(i, J)}^{(a_1 \cdot w, B)}, x_i] = a_1 \cdot w \partial_J^B = a_1 \left(\partial_J^{w \cdot B} + \sum_{k=1}^n \partial_{(k, J)}^{([w, x_k], B)} \right)$$

(proof of Proposition 3.7);

$$[\partial_{(i, J)}^{(a_1 \cdot w, B)}, x_t] = 0 \quad \text{for } t \neq i, \quad \text{and} \quad \partial_{(i, J)}^{(a_1 \cdot w, B)}(1) = 0.$$

$$\text{Hence, } \partial_{(i, J)}^{(a_1 \cdot w, B)} = \partial_{(i, J)}^{(a_1, w \cdot B)} + \sum_{k=1}^n \partial_{(i, k, J)}^{(a_1, [w, x_k], B)}.$$

Now, for $r > 1$ and using induction,

$$[\partial_{(I, J)}^{(A \cdot w, B)} - \partial_{(I, J)}^{(A, w \cdot B)}, x_t] = \left[\sum_{k=1}^n \partial_{(I, k, J)}^{(A, [w, x_k], B)}, x_t \right] \quad \text{for } t \leq n, \text{ and}$$

$$\partial_{(I, J)}^{(A \cdot w, B)} - \partial_{(I, J)}^{(A, w \cdot B)}(1) = 0.$$

Hence the result. □

Remark 3.8. *For any indexing tuple $I = (i_1, \dots, i_r)$, and a word $w \in R$, denote by w_I , the tuple $(1, \dots, 1, w)$. The above proposition implies that every $\varphi \in D_{\mathbb{k}}(R)$ can be written in the form of a series*

$$\varphi = \lambda_a + \sum_{I, w} \alpha_{I, w} \partial_I^{w_I} \quad \text{for } \alpha_{I, w} \in \mathbb{k}, a \in R, \text{ and words } w \in R.$$

But we have the following finite and unique description for any $\varphi \in D_{\mathbb{k}}(R)$ noting that $D_{\mathbb{k}}(R)$ as a vector space has a spanning set as given by Proposition 3.7.

Corollary 3.5. *Let $r \geq 0$ be the minimum natural number for which φ is in the span of the set $\{\lambda_a \mid a \text{ is a word in } R\} \cup \{\partial_I^A \mid |I| \leq r\}$. Then for finitely many words $w \in R$ and finitely many $A = (a_1, \dots, a_r)$, $a, a_l \in R$, φ can be uniquely written*

$$\varphi = \lambda_a + \sum_{|I| < r, w} \alpha_{I,w} \partial_I^{w_I} + \sum_{|I|=r, A} \alpha_{I,A} \partial_I^A.$$

Proof. By Propositions 3.7 and 3.8 we see that φ can indeed be written in the given fashion. For uniqueness, suppose

$$\varphi = \lambda_a + \sum_{|I| < r, w} \alpha_{I,w} \partial_I^{w_I} + \sum_{|I|=r, A} \alpha_{I,A} \partial_I^A = 0$$

for finitely many words $w \in R$, and finitely many $A = (a_1, \dots, a_r)$, with $a, a_l \in R$. Then, $\varphi(1) = 0$ implies $a = 0$. Since, for each $I = (i_1, \dots, i_t)$ with $t < r$, $\partial_I^{w_I}(x_{i_1} \cdots x_{i_t}) = w$, we have $\varphi(x_{i_1} \cdots x_{i_t}) = \alpha_{I,w} = 0$. Thus, $\varphi = \sum_{|I|=r, A} \alpha_{I,A} \partial_I^A = 0$. Now we refer to Proposition 3.7 to claim that $\alpha_{I,A} = 0$. \square

Remark 3.9. *The proof of the above Corollary also proves that the series description as in the Remark 3.8 is also unique. We say that φ is written in the **finite canonical form** (respectively, the **power series canonical form**) if it is written as described in the above corollary (respectively, remark 3.8).*

Theorem 3.4. *The algebra $D_{\mathbb{k}}(R)$ is simple.*

Proof. Suppose I is a nonzero ideal of $D(R)$. Let $\varphi \in I$ be written in the finite canonical form. Note that $[\varphi, \partial_i^1]$ is also in the finite canonical form with reduced degree. Thus, taking successive commutators with appropriate ∂_i^1 we can assume that $\varphi = \lambda_1 + \sum_{|I| \leq r} \alpha_I \partial_I^1$ where $\mathbf{1} = (1, 1, \dots, 1)$, unless, $[\varphi, \partial_i^1] = 0 \forall i$. In the former case, we can take appropriate commutators with x_i to arrive at $1 \in I$. In the latter case, we refer to part (2) of the Proposition 3.4, and consider $[\varphi, \partial_{(i,j)}^{(1,1)}] \in I \forall i, j$. Taking commutator with $\partial_{(i,j)}^{(1,1)}$ results in an operator of reduced degree written in the finite canonical form, unless the commutator is 0 $\forall i, j$. In the latter case, we take commutators with appropriate $\partial_{(i,j,k)}^{(1,1,1)}$. Continuing thus we arrive at the result. \square

Remark 3.10. *We believe that the canonical forms can be used to prove that $D_{\mathbb{k}}(R)$ is a domain when characteristic of \mathbb{k} is zero. We are unable to prove the same.*

We also believe that the algebra generated by $D_{\mathbb{k}}^k(R)$ is not all of $D_{\mathbb{k}}(R)$. We have seen in Remark 3.3 that $\Delta(R) \neq D_{\mathbb{k}}(R)$. We are not able to prove that, $D_{\mathbb{k}}(R)$ is not finitely generated. Conjecturally, even when the characteristic of \mathbb{k} is zero, $\langle D_{\mathbb{k}}^k(R) \rangle \neq D_{\mathbb{k}}(R)$ for any $k \geq 0$.

For $A = \mathbb{k}[y_1, \dots, y_n]$ the polynomial algebra over a field \mathbb{k} of characteristic 0 in n variables, and I be an ideal of A , we have $\mathcal{S}_I/\mathcal{J}_I \cong D_{\mathbb{k}}(A/I)$ as filtered algebras. Further, $\mathcal{J}_I = ID_{\mathbb{k}}(A)$ and \mathcal{S}_I is the ring generated by \mathcal{J}_I (see chapter 15 and the references given in section 15.6 of [7]). We would like to see an analogue of these statements for the free algebra $R = \mathbb{k}\langle x_1, \dots, x_n \rangle$, $n > 1$, with a two sided ideal I .

Proposition 3.9. *The natural map $\mathcal{S}_I/\mathcal{J}_I \rightarrow D_{\mathbb{k}}(R/I)$ of filtered algebras is surjective.*

Proof. Let $\bar{a} \in R/I$ denote the image of $a \in R$ and $\bar{\varphi} \in D_{\mathbb{k}}(R/I)$ denote the image of $\varphi \in \mathcal{S}_I$. Note $\lambda_{\bar{a}} = \overline{\lambda_a}$ and $\rho_{\bar{a}} = \overline{\rho_a}$. Hence $D_{\mathbb{k}}^0(R/I)$ is in the image of $\mathcal{S}_I/\mathcal{J}_I$.

Suppose $\eta \in D_{\mathbb{k}}^t(R/I)$ is such that $[\eta, \bar{x}_i] = \bar{\psi}_i \in D_{\mathbb{k}}^{t-1}(R/I)$ for $t \geq 1$, and $i \leq n$ and $\eta(\bar{1}) = \bar{0}$. Let $\varphi \in D_{\mathbb{k}}^t(R)$ be such that $[\varphi, x_i] = \psi_i \in D_{\mathbb{k}}^{t-1}(R)$ and $\varphi(1) = 0$. Then $\bar{\varphi} = \eta$ (and therefore $\varphi \in \mathcal{S}_I$). \square

4. β -DIFFERENTIAL OPERATORS ON THE \mathbb{Z}^n -GRADED FREE ALGEBRA.

Let $R = \mathbb{k}\langle x_1, \dots, x_n \rangle$ be the free \mathbb{k} -algebra generated by variables x_1, \dots, x_n . The algebra R is \mathbb{Z}^n graded by setting degree of x_i to be $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ where 1 appears in the i -th place. For $i, j \leq n$, let $q_{ij} \in \mathbb{k}^*$. We define $\beta : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{k}^*$ by setting $\beta(e_i, e_j) = q_{ij}$. In case $q_{ij}q_{ji} = 1$ for $i \neq j$, we have $\beta(a, b)\beta(b, a) = 1$ for $a \neq b$ which implies that if φ_1, φ_2 are two left β -derivations, then so is $[\varphi_1, \varphi_2]_{\beta}$.

For each homogeneous $a \in R$, and each $i \leq n$, there is a left β -derivation $\partial_{\beta, i}^a$ such that $\partial_{\beta, i}^a(x_j) = \delta_{i, j}a$. Note that $d_{\partial_{\beta, i}^a} = d_a - e_i$ (recall the notation, d_m = degree of m). Moreover, for $a \in R$, we have the β -inner derivation,

$$\lambda_a - \rho_a^{\beta} = \sum_i \partial_{\beta, i}^{[a, x_i]_{\beta}}.$$

Remark 4.1. *In the case when $q_{ij}q_{ji} = q_{ii} = 1$, the vector space of β -derivations on R , denoted $\text{Der}_{\beta}(R)$, is a coloured Lie algebra, which is not simple. There is a surjection from $\text{Der}_{\beta}(R)$ to $\text{Der}_{\beta}(\bar{R})$ where \bar{R} is the quotient algebra of R subject to the relations $x_i x_j = q_{ij} x_j x_i$ for $i, j \leq n$. The β -inner derivations are in the kernel of this surjection.*

Definition 4.1. *For $r = 1, I = (i_1), i_1 \leq t$, and $A = (a_1)$, for homogeneous $a_1 \in R$ set $\partial_{\beta, I}^A = \partial_{\beta, i_1}^{a_1}$. For an $r \in \mathbb{N}, r \geq 2$, let $I = (i_1, i_2, \dots, i_r)$ be a sequence of natural numbers $i_j \leq t$ and $A = (a_1, \dots, a_r)$ be a sequence of homogeneous elements from R . Further, let $\hat{I} = (i_2, \dots, i_r)$ and $\hat{A} = (a_2, \dots, a_r)$. Denote by $\partial_{\beta, I}^A \in D_{\beta}^r(R)$ the operator which satisfies the commutator rules*

$$[\partial_{\beta, I}^A, x_{i_1}]_{\beta} = a_1 \partial_{\beta, \hat{I}}^{\hat{A}}, \quad [\partial_{\beta, I}^A, x_l]_{\beta} = 0 \text{ for } l \neq i_1, \text{ and } \quad \partial_{\beta, I}^A(1) = 0.$$

Let $\Delta_{\beta}(R)$ denote the associative subalgebra of $D_{\beta}(R)$ generated by $D_{\beta}^1(R)$ and $\Delta_{\beta}^r(R) = \Delta_{\beta}(R) \cap D_{\beta}^r(R)$ for $r \geq 1$.

The proofs of the items in the following theorem are similar to those in the section 3.

- Theorem 4.1.** (1) *The associative, \mathbb{Z}^n -graded algebras $R \otimes^\beta R^{\beta o}$ and $D_\beta^0(R)$ are isomorphic.*
 (2) *The β -centre of $D_\beta(R)$ is \mathbb{k} .*
 (3) *A β -derivation of R which is in $D_\beta^0(R)$ is a sum of inner β -derivations.*
 (4) *For homogeneous $a, b \in R$ and i, j between 1 and n , we have*

$$[\partial_{\beta,i}^a, \lambda_b]_\beta = \lambda_{\partial_{\beta,i}^a(b)}$$

Further, when $q_{ij}q_{ji} = 1$ and $q_{ii} = 1$ we have,

$$[\partial_{\beta,i}^a, \rho_b^\beta]_\beta = \rho_{\partial_{\beta,i}^a(b)}^\beta$$

and

$$[\partial_{\beta,i}^a, \partial_{\beta,j}^b]_\beta = \partial_{\beta,j}^{\partial_{\beta,i}^a(b)} - \beta(d_a - e_i, d_b - e_j) \partial_{\beta,i}^{\partial_{\beta,j}^b(a)}.$$

- (5) *Let $r \geq 1$ and let I and A be sequences $I = (i_1, \dots, i_r)$ and $A = (a_1, \dots, a_r)$ with i_t between 1 and n and $a_t \in R$.*

$$\partial_{\beta,I}^A(x_{t_1} \cdots x_{t_r}) = \begin{cases} a_1 \cdots a_r & \text{if } (t_1, \dots, t_r) = I \\ 0 & \text{if } (t_1, \dots, t_r) \neq I \end{cases}$$

Further, $\partial_{\beta,I}^A(x_{t_1} \cdots x_{t_k}) = 0$ for $k < r$.

- (6) $[\partial_{\beta,(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}, \lambda_a]_\beta = \sum_{j=2}^r \alpha_j \lambda_{\partial_{\beta,(i_1, \dots, i_{j-1})}^{(a_1, \dots, a_{j-1})}(a)} \partial_{\beta,(i_j, \dots, i_r)}^{(a_j, \dots, a_r)}$ for $r \geq 2$, where
 $\alpha_j = \beta(d_{a_j \cdots a_r} - e_{i_j} - \cdots - e_{i_r}, d_a - e_{i_1} - \cdots - e_{i_j}).$

This generalizes the case of $r = 1$ from above: $[\partial_{\beta,i}^a, \lambda_b]_\beta = \lambda_{\partial_{\beta,i}^a(b)}$. In particular, $[\partial_{\beta,(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}, \lambda_a]_\beta \in D_\beta^{r-1}(R)$.

- (7) $[\partial_{\beta,(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}, \rho_a^\beta]_\beta = \sum_{j=2}^r \alpha_j \rho_{\partial_{\beta,(i_1, \dots, i_{j-1})}^{(a_1, \dots, a_{j-1})}(a)}^\beta \partial_{\beta,(i_j, \dots, i_r)}^{(a_j, \dots, a_r)}$ when $r \geq 2$,

$q_{ij}q_{ji} = 1$, $q_{ii} = 1$ and

$$\alpha_j = \beta(d_{a_1 \cdots a_{j-1}} - e_{i_1} - \cdots - e_{i_{j-1}}, d_b) \beta(d_{a_1 \cdots a_{j-1}}, d_{a_j \cdots a_r} - e_j - \cdots - e_r).$$

This generalizes the case of $r = 1$ from above: $[\partial_{\beta,i}^a, \rho_b^\beta]_\beta = \rho_{\partial_{\beta,i}^a(b)}^\beta$. In particular, $[\partial_{\beta,(i_1, \dots, i_r)}^{(a_1, \dots, a_r)}, \rho_a^\beta]_\beta \in D_\beta^{r-1}(R)$.

(8)

$$\partial_{\beta,I}^A(ab) = \sum_{(J,K)=I} \alpha_{J,K} \partial_{\beta,J}^{A_J}(a) \partial_{\beta,K}^{A_K}(b)$$

where $\alpha_{J,K} = \beta(\deg(\partial_{\beta,K}^A), \deg(\partial_{\beta,J}^{A_J}))$.

- (9) *Let $J = (j_1, \dots, j_s)$ and $B = (b_1, \dots, b_s)$. Then there are scalars α_τ such that*

$$\partial_{\beta,I}^A \partial_{\beta,J}^B = \sum_{\tau \in T_{(r,s)}} \alpha_\tau \partial_{\beta,\tau((I,J))}^{\tau((A,B))} + \text{terms of lower order}$$

Recall that $T_{(r,s)}$ consists of $\tau \in S_{r+s}$ such that τ is increasing on $1, \dots, r$ and on $r+1, \dots, r+s$.

- (10) When $q_{ij}q_{ji} = 1$ and $q_{ii} = 1$ for all i and j , then for $\partial_{\beta,I}^A \in D_{\beta}^r$ and $\partial_{\beta,J}^B \in D_{\beta}^s$, we have $[\partial_{\beta,I}^A, \partial_{\beta,J}^B]_{\beta} \in D_{\beta}^{r+s-1}$.
- (11) We have the formula

$$\partial_{\beta,(I,J)}^{(A \cdot w, B)} - \partial_{\beta,(I,J)}^{(A, w \cdot B)} = \sum_{k=1}^n \alpha_k \partial_{\beta,(I,k,J)}^{(A, [w, x_k]_{\beta}, B)}$$

where $\alpha_k = \beta(\deg(\partial_{\beta,J}^B), \deg(x_k))$.

- (12) For $r \geq 1$, Then there are scalars α_{σ} such that

$$\sum_{\sigma \in S_r} \alpha_{\sigma} \partial_{\beta, \sigma(I)}^{\sigma(A)} \in \Delta_{\beta}(R)$$

where S_r is the permutation group on r symbols.

- (13) $\Delta_{\beta}(R) \neq D_{\beta}(R)$
- (14) When $q_{ij}q_{ji} = 1$ and $q_{ii} = 1$, the algebra $\Delta_{\beta}(R)$ is not simple.
- (15) As a left $D_{\beta}^0(R)$ -module, $D_{\beta}^r(R)$ is generated by the set

$$\{\partial_{\beta,I}^A \mid I = (i_1, \dots, i_r), A = (a_1, \dots, a_r)\} \cup \{\lambda_1\}.$$

- (16) The set $\{\partial_{\beta,I}^A \mid I = (i_1, \dots, i_r), A = (a_1, \dots, a_r) \text{ } a_i \text{ is a word in } R\}$ generates a free $D_{\beta}^0(R)$ -submodule of $D_{\beta}^r(R)$.
- (17) As a K -vector space, $D_{\beta}(R)$ is spanned by the set

$$\begin{aligned} & \{\partial_{\beta,I}^A \mid I = (i_1, \dots, i_r), A = (a_1, \dots, a_r) \text{ where each } a_i \text{ is a word in } R\} \\ & \cup \{\lambda_a \mid a \text{ is a word in } R\} \end{aligned}$$

- (18) The algebra $D_{\beta}(R)$ is simple when $q_{ij} = q_{ji}$ and $q_{ii} = 1$.
- (19) The map described in Proposition 2.2 is surjective when R is free.

5. QUANTUM DIFFERENTIAL OPERATORS ON THE \mathbb{Z}^n -GRADED FREE ALGEBRA.

As in the previous section, let $R = \mathbb{k} \langle x_1, \dots, x_n \rangle$ be the free \mathbb{k} -algebra generated by variables x_1, \dots, x_n . The algebra R is \mathbb{Z}^n graded by setting degree of x_i to be e_i , and we define $\beta : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{k}^*$ by setting $\beta(e_i, e_j) = q_{ij}$ for fixed $q_{ij} \in \mathbb{k}^*$.

For each $\gamma \in \Gamma$, recall $\sigma_{\gamma} \in D_q^0(R)$ given by $\sigma_{\gamma}(r) = \beta(\gamma, d_r)r$ for homogeneous r and extended linearly, the grading map. Let $\Lambda \subset \text{Aut}_{\mathbb{k}}(R)$ be the subgroup generated by $\{\sigma_{\gamma} \mid \gamma \in \Gamma\}$. Then we have a surjection $(R \otimes R^o) \# \Lambda \rightarrow D_q^0(R)$ (recall from Section 2 the surjection $(R \otimes R^o) \# \Gamma \rightarrow D_q^o(R)$). Note that for any homogeneous $\varphi \in \text{grHom}_{\mathbb{k}}(R, R)$ of degree d_{φ} , we have

$$\sigma_{\gamma}\varphi = \beta(\gamma, d_{\varphi})\varphi\sigma_{\gamma}.$$

Hence, every $\varphi \in D_q^0(R)$ can be written as a finite sum, $\varphi = \sum_{\sigma \in \Lambda} \psi_{\sigma}\sigma$ for $\psi_{\sigma} \in D_{\mathbb{k}}^0(R)$.

For homogeneous φ, ψ , let $[\varphi, \psi]_\gamma = \varphi\psi - \beta(\gamma, d_\psi)\psi\varphi$. The *quantum centre* of $D_q(R)$ is the subalgebra generated by the set $\{\text{homogeneous } \varphi \in D_q(R) \mid \exists \gamma \in \Gamma \text{ such that } [\varphi, \psi]_\gamma = 0 \forall \psi \in D_q(R)\}$.

Proposition 5.1. *The quantum centre of $D_q(R)$ is \mathbb{k} .*

Proof. Let homogeneous φ be in the quantum centre of $D_q(R)$. Then there exists a $\gamma \in \Gamma$ such that $[\varphi, \psi]_\gamma = 0, \forall \psi \in D_q(R)$. In particular, $[\varphi, \lambda_r]_\gamma = 0, \forall r \in R$. Hence, $\varphi = \rho_{\varphi(1)}\sigma_\gamma$. Hence, for any $r \in R$, $[\varphi, \rho_r]_\gamma = \rho_{[\varphi(1), \sigma_\gamma(r)]}\sigma_\gamma$. Thus, $\varphi(1)$ is in the usual centre of R . \square

Definition 5.1. For each $i \leq n, \gamma \in \Gamma$, and $a \in R$, denote by $\partial_{i,\gamma}^a \in D_q^1(R)$ defined by $[\partial_{i,\gamma}^a, x_j] = \delta_{i,j}\lambda_a\sigma_\gamma$ and $\partial_{i,\gamma}^a(1) = 0$.

For each natural number $r \geq 1$, let $I = (i_1, i_2, \dots, i_r)$, $K = (\gamma_1, \gamma_2, \dots, \gamma_r)$, and $A = (a_1, \dots, a_r)$ for $1 \leq i_1, \dots, i_r \leq n$, $\gamma_1, \dots, \gamma_r \in \Gamma$, and $a_1, \dots, a_r \in R$.

When $r = 1$, let $\partial_{I,K}^A = \partial_{i_1,\gamma_1}^{a_1}$.

For $r \geq 2$, let $\partial_{I,K}^A \in D_q^r(R)$ be that operator defined by

$$[\partial_{I,K}^A, x_j] = \delta_{i_1,j}a_1\partial_{\widehat{I},\widehat{K}}^{\widehat{A}}\sigma_{\gamma_1}, \quad \partial_{I,K}^A(1) = 0,$$

where $\widehat{I} = (i_2, \dots, i_r)$, $\widehat{K} = (\gamma_2, \dots, \gamma_r)$, and $\widehat{A} = (a_2, \dots, a_r)$.

Proposition 5.2. *For each $i \leq n$, $a \in R$, and $\gamma \in \Gamma$, the operator $\partial_{i,\gamma}^a$ is a right skew σ_γ -derivation.*

Proof. We need to prove that $[\partial_{i,\gamma}^a, r] = \lambda_{\partial_{i,\gamma}^a(r)}\sigma_\gamma \forall r \in R$. We know that this is true for $r \in \{x_1, \dots, x_n, 1\}$. Assume that the proposition is true for every word of length less than k . Suppose that rs is a word of length k . Then,

$$\begin{aligned} [\partial_{i,\gamma}^a, rs] &= [\partial_{i,\gamma}^a, r]s + r[\partial_{i,\gamma}^a, s] = \lambda_{\partial_{i,\gamma}^a(r)}\sigma_\gamma s + r\partial_{i,\gamma}^a(s)\sigma_\gamma \\ &= \lambda_{\partial_{i,\gamma}^a(r)}\sigma_\gamma(s)\sigma_\gamma + r\partial_{i,\gamma}^a(s)\sigma_\gamma = (\partial_{i,\gamma}^a(r)\sigma_\gamma(s) + r\partial_{i,\gamma}^a(s))\sigma_\gamma \end{aligned}$$

Now, $\partial_{i,\gamma}^a(rs) = \partial_{i,\gamma}^a(rs1) = rs\partial_{i,\gamma}^a(1) + [\partial_{i,\gamma}^a, rs](1) = \partial_{i,\gamma}^a(r)\sigma_\gamma(s) + r\partial_{i,\gamma}^a(s)$. \square

Remark 5.1. *The above proposition shows that*

$$\partial_{i,\gamma}^a\lambda_r - \lambda_r\partial_{i,\gamma}^a = \lambda_{\partial_{i,\gamma}^a(r)}\sigma_\gamma \quad \text{and} \quad \partial_{i,\gamma}^a\rho_s - \rho_{\sigma_\gamma(s)}\partial_{i,\gamma}^a = \rho_{\partial_{i,\gamma}^a(s)}$$

We see that $D_q^1(R)$ is generated as a module over $D_q^0(R)$ by the right σ_γ -derivations $\partial_{i,\gamma}^a$. Recall that φ is a left skew σ -derivation, if and only if $\varphi\sigma^{-1}$ is a right skew σ^{-1} -derivation.

Let $\Delta_q(R)$ be the subalgebra of $\text{grHom}_{\mathbb{k}}(R, R)$ generated by $D_q^0(R)$ and the operators $\partial_{i,\gamma}^a$ for $i \leq n, a \in R$ and $\gamma \in \Gamma$.

For any $a \in R$, and $\gamma \in \Gamma$, the operator $\lambda_a - \rho_a\sigma_\gamma$ is a left skew- σ_γ derivation. We call such a left skew- σ_γ derivations, an inner left- σ_γ derivation. Following the same proof as in Proposition 3.2, we see that any left skew- σ_γ derivation of R which is in $D_q^0(R)$ is a sum of inner left- σ_γ derivations.

Similarly, $\lambda_a \sigma_\gamma - \rho_a$ is a right skew- σ_γ derivation. Such a right skew- σ_γ derivations is called, an inner right- σ_γ derivation. Any right skew- σ_γ derivation of R which is in $D_q^0(R)$ is a sum of inner right- σ_γ derivations.

Here are some more generalizations of the usual and β differential operators. The proofs are similar to the corresponding ones in the section 3.

Theorem 5.1. Let $I = (i_1, \dots, i_r)$, $A = (a_1, \dots, a_r)$, and $K = (\gamma_1, \dots, \gamma_r)$ as in the definition above.

(1)

$$\partial_{I,K}^A(x_{t_1} x_{t_2} \cdots x_{t_r}) = \begin{cases} \left(\prod_{1 \leq j < s \leq r} \beta(\gamma_j, e_{i_s}) \right) a_1 a_2 \cdots a_r & \text{if } I = (t_1, t_2, \dots, t_r) \\ 0 & \text{if } I \neq (t_1, t_2, \dots, t_r) \end{cases}$$

Further, $\partial_{I,K}^A(x_{t_1} x_{t_2} \cdots x_{t_k}) = 0$ for $k < r$.

(2) Let $J = (i_{j_1}, \dots, i_{j_t})$ be a subsequence of I . Let $A_J = (a_{j_1}, \dots, a_{j_t})$ and $K_J = (\gamma_{j_1}, \dots, \gamma_{j_t})$ respectively represent the corresponding subsequences of A and K . Further, let $\sigma_{K_J} = \sigma_{\gamma_{j_1}} + \cdots + \sigma_{\gamma_{j_t}}$. Then, for $a, b \in R$,

$$\partial_{I,K}^A(a \cdot b) = \sum_{(I_1, I_2)=I} \partial_{I_1, K_{I_1}}^{A_{I_1}}(a) \cdot \partial_{I_2, K_{I_2}}^{A_{I_2}}(\sigma_{K_{I_1}}(b))$$

where (I_1, I_2) denotes the concatenation of I_1 and I_2 . Therefore, (recall Remark 5.1)

$$\partial_{I,K}^A \lambda_a - \lambda_a \partial_{I,K}^A = \sum_{(I_1, I_2)=I, I_1, I_2 \neq \{\}} \lambda_{\partial_{I_1, K_{I_1}}^{A_{I_1}}(a)} \partial_{I_2, K_{I_2}}^{A_{I_2}} \sigma_{K_{I_1}}$$

$$\partial_{I,K}^A \rho_b - \rho_{\sigma_K(b)} \partial_{I,K}^A = \sum_{(I_1, I_2)=I, I_1, I_2 \neq \{\}} \rho_{\partial_{I_2, K_{I_2}}^{A_{I_2}}(\sigma_{K_{I_1}}(b))} \partial_{I_1, K_{I_1}}^{A_{I_1}}.$$

(3) Let S_r denote the permutation group on r symbols, and for any $\tau \in S_r$ let $\tau(I) = (i_{\tau(1)}, i_{\tau(2)}, \dots, i_{\tau(r)})$, and similarly define $\tau(A), \tau(K)$. Then, for each $\tau \in S_r$ there exists $\alpha_\tau \in \mathbb{k}$ such that

$$\sum_{\tau \in S_r} \alpha_\tau \partial_{\tau(I), \tau(K)}^{\tau(A)} \in \Delta_q(R).$$

In fact, $\sum_{\tau \in S_r} \alpha_\tau \partial_{\tau(I), \tau(K)}^{\tau(A)} - \partial_{i_1, \gamma_1}^{a_1} \partial_{i_2, \gamma_2}^{a_2} \cdots \partial_{i_r, \gamma_r}^{a_r} \in \Delta_q(R) \cap D_q^{r-1}(R)$. Moreover, if d_m denotes the degree of the operator $\partial_{i_m, \gamma_m}^{a_m}$ for $1 \leq m \leq r$, then the scalar

$$\alpha_\tau = \prod_{1 \leq \tau(m) < \tau(n) \leq r} \beta(\gamma_{\tau(m)}, d_{\tau(n)}).$$

(4) Let $J = (j_1, \dots, j_s)$, $B = (b_1, \dots, b_s)$, and $L = (\delta_1, \dots, \delta_s)$. Then there are scalars α_τ such that

$$\partial_{I,K}^A \partial_{J,L}^B = \sum_{\tau \in T_{(r,s)}} \alpha_\tau \partial_{\tau(I), \tau(K)}^{\tau(A,B)} + \text{terms of lower order}$$

Recall that $T_{(r,s)}$ consists of $\tau \in S_{r+s}$ such that τ is increasing on $1, \dots, r$ and on $r+1, \dots, r+s$. For $1 \leq m \leq r+s$, let $d_m = \text{degree of } \partial_{i_m, \gamma_m}^{a_m}$ and $\eta_m = \gamma_m$ for $m \leq r$ and $d_m = \text{degree of } \partial_{j_{m-r}, \delta_{m-r}}^{b_{m-r}}$ for $m > r$ and $\eta_m = \delta_{m-r}$. Then

$$\alpha_\tau = \prod_{1 \leq \tau(m) < \tau(n) \leq r+s} \beta(\eta_{\tau(m)}, d_{\tau(n)}).$$

(5) We have the following formula

$$a \partial_{I,K}^A - \partial_{I,K}^{a \cdot A} = \sum_{k=1}^n \partial_{(k,I),(0,K)}^{([a, x_k], A)};$$

here, $(a \cdot A) = (aa_1, a_2, \dots, a_r)$, $(k, I) = (k, i_1, \dots, i_r)$, $(0, K) = (0, \gamma_1, \dots, \gamma_r)$, and $([a, x_k], A) = ([a, x_k], a_1, \dots, a_r)$.

With the same notations as used above, or in previous sections,

$$\partial_{(I,J),(K,L)}^{(A \cdot w, B)} - \partial_{(I,J),(K,L)}^{(A, w \cdot B)} = \sum_{k=1}^n \partial_{(I,k,J),(K,0,L)}^{(A, [w, x_k], B)}$$

(6) As a left $D_q^0(R)$ -module, $D_q^r(R)$ is generated by the set

$$\{\partial_{I,K}^A \mid I = (i_1, \dots, i_r), K = (\gamma_1, \dots, \gamma_r), A = (a_1, \dots, a_r)\} \cup \{\lambda_1\}.$$

(7) The map described in Proposition 2.3 is surjective when R is free.

Remark 5.2. (1) Note that $D_{\mathbb{k}}(R) \subset D_q(R)$ and we have a map

$$D_{\mathbb{k}}(R) \# \Lambda \rightarrow D_q(R).$$

In general, this map need not be surjective. Indeed, in [3] the generators of the algebra $D_q(\mathbb{k}[x])$ have been described over a field of characteristic 0. The polynomial algebra $\mathbb{k}[x]$ is \mathbb{Z} -graded, and $\beta : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{k}^*$ is given by $\beta(n, m) = q^{nm}$ for q a transcendental element over \mathbb{Q} , and $\mathbb{Q}(q) \subset \mathbb{k}$. Then $D_q(\mathbb{k}[x])$ is generated by the set $\{\lambda_x, \partial^\beta, \partial, \partial^{\beta^{-1}}\}$, where $\partial(x^n) = nx^{n-1}$, $\partial^\beta(x^n) = \left(\frac{q^n - 1}{q - 1}\right)x^{n-1}$, $\partial^{\beta^{-1}}(x^n) = \left(\frac{q^{-n} - 1}{q^{-1} - 1}\right)x^{n-1}$, for $n \geq 1$ and $\partial(1) = \partial^\beta(1) = \partial^{\beta^{-1}}(1) = 0$. The algebra $D_{\mathbb{k}}(\mathbb{k}[x])$ is the first Weyl algebra, with generators $\{\lambda_x, \partial\}$. We see that $\partial^\beta \notin D_{\mathbb{k}}(\mathbb{k}[x]) \# \mathbb{Z}$. Indeed, by degree considerations, if $\partial^\beta \in D_{\mathbb{k}}(\mathbb{k}[x]) \# \mathbb{Z}$, then $\partial^\beta = \alpha \partial \sigma$ for some $\alpha \in \mathbb{k}$, which is not possible.

(2) Corollary 3.4 and part (10) of Theorem 4.1 do not generalize to the case of quantum differential operators. For example, consider $D_q(\mathbb{k} \langle x_1, x_2 \rangle)$ when $q_{11} = q_{22} = 1$ and $q_{21} = q_{12}^{-1} = q$ for $q \in \mathbb{k}^*$, where q is transcendental over \mathbb{Q} , and $\mathbb{Q}(q) \subset \mathbb{k}$. Note that $\partial_{1,e_1}^1, \partial_{1,e_2}^1 \in D_q^1(\mathbb{k} \langle x_1, x_2 \rangle)$, with $\partial_{1,e_1}^1(x_1^n) = nx_1^{n-1}$, while $\partial_{1,e_2}^1(x_1^n) = (1 + q + \dots, q^{n-1})x_1^{n-1}$ for $n \geq 1$. But $[\partial_{1,e_1}^1, \partial_{1,e_2}^1]_\gamma \notin D_q^0(\mathbb{k} \langle x_1, x_2 \rangle)$ for any $\gamma \in \Gamma$, which can be seen by degree considerations.

- (3) *The question whether the algebra $D_q(R)$ is simple seems to be a difficult question to address. Conjecturally, we believe that $D_q(R)$ is simple. The algebra $\Delta_q(R)$ we expect to be not simple, as has been already checked when $\beta \equiv 1$.*

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